

# Geometric configurations, regular subalgebras of $E_{10}$ and M-theory cosmology

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ABSTRACT: We re-examine previously found cosmological solutions to eleven-dimensional supergravity in the light of the  $E_{10}$ -approach to M-theory. We focus on the solutions with non zero electric field determined by geometric configurations  $(n_m, g_3)$ ,  $n \leq 10$ . We show that these solutions are associated with rank  $g$  regular subalgebras of  $E_{10}$ , the Dynkin diagrams of which are the (line) incidence diagrams of the geometric configurations. Our analysis provides as a byproduct an interesting class of rank-10 Coxeter subgroups of the Weyl group of  $E_{10}$ .

KEYWORDS: M-Theory, String Duality, Global Symmetries.

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## 1. Introduction

Even in the cosmological context of homogeneous fields  $G_{\alpha\beta}(t)$ ,  $F_{\alpha\beta\gamma\delta}(t)$  that depend only on time (“Bianchi I cosmological models” [1]), the equations of motion of eleven-dimensional supergravity remain notoriously complicated. This is because the dynamical behavior of the system is, for generic initial conditions, a never ending succession of “free” Kasner regimes interrupted by “collisions” against “symmetry” or “electric” walls [2, 3]. During a given Kasner regime, the energy-momentum of the 3-form potential can be neglected and the scale factors of the spatial metric have the typical Kasner power law behavior  $\sim t^{2p_i}$  with  $\sum_i p_i = 1 = \sum_i p_i^2$  in terms of the proper time  $t$ . Any of these free flight motions ultimately ends with a collision, leading to a transition to a new Kasner regime characterized by new Kasner exponents. In the collision against an electric wall, the energy density of the electric field becomes comparable to the Ricci tensor for the short time of duration of the collision. The localization in time of the collision and hence of the corresponding electric energy density gets sharper and sharper as one goes to the cosmological singularity<sup>1</sup>. The model is a simple example exhibiting the intricate BKL-type phenomenon [4].

The dynamical behavior of the system can be represented as a billiard motion [5–8, 3, 9] in the fundamental Weyl chamber of the Lorentzian Kac-Moody algebra  $E_{10}$  [3]. The hyperbolic character of  $E_{10}$  accounts for the chaotic properties of the dynamics [10].

In order to get a more tractable dynamical system, one may impose further conditions on the metric and the 4-form. This must be done in a manner compatible with the equations of motion: if the additional conditions are imposed initially, they should be preserved by the time evolution. One such set of conditions is that the spatial metric be diagonal,

$$ds^2 = -N^2(x^0)(dx^0)^2 + \sum_{i=1}^{10} a_i^2(x^0)(dx^i)^2. \tag{1.1}$$

Invariance under the ten distinct spatial reflections  $\{x^j \rightarrow -x^j, x^{i \neq j} \rightarrow x^{i \neq j}\}$  of the metric is compatible with the Einstein equations only if the energy-momentum tensor of the 4-form is also diagonal. Although one cannot impose on the 4-form itself to be reflection

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<sup>1</sup>We are simplifying the discussion by assuming the collisions to be clearly separated in time. In general, “multiple collisions” might take place, without changing the qualitative picture [2]. Also, when the magnetic field is non zero, there can be collisions against magnetic walls.

invariant without forcing it to vanish, one can ensure that the energy-momentum tensor is reflection invariant.

A large class of electric solutions to the question of finding  $F_{\alpha\beta\gamma\delta}$  such that  $T_{\mu\nu}$  is invariant under spatial reflections was found in [1]. These can be elegantly expressed in terms of geometric configurations  $(n_m, g_3)$  of  $n$  points and  $g$  lines (with  $n \leq 10$ ). That is, for each geometric configuration  $(n_m, g_3)$  (whose definition is recalled below), one can associate diagonal solutions with some non-zero electric field components  $F_{0ijk}$  determined by the configuration. The purpose of this paper is to re-examine this result in the light of the attempt to reformulate M-theory as an  $E_{10}$  non linear  $\sigma$ -model in one dimension.

It has recently been shown in [11] that the dynamical equations of eleven-dimensional supergravity can be reformulated as the equations of motion of the one-dimensional non linear  $\sigma$ -model  $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ , where  $\mathcal{K}(\mathcal{E}_{10})$  is the subgroup of  $\mathcal{E}_{10} \equiv \exp E_{10}$  obtained by exponentiating the subalgebra  $K(E_{10})$  of  $E_{10}$  invariant under the standard Chevalley involution<sup>2</sup>. Although the matching works only at low levels (with the dictionary between the two theories derived so far), it provides further intriguing evidence that infinite dimensional algebras of  $E$ -type might underlie the dynamics of M-theory [12, 3, 13].

We prove here that the conditions on the electric field embodied in the geometric configurations  $(n_m, g_3)$  have a direct Lie-algebraic interpretation. They simply amount to consistently truncating the  $E_{10}$  non linear  $\sigma$ -model to a  $\bar{\mathfrak{g}}$  non linear  $\sigma$ -model, where  $\bar{\mathfrak{g}}$  is a rank- $g$  Kac-Moody subalgebra of  $E_{10}$  (or a quotient of such a Kac-Moody subalgebra by an appropriate ideal when the relevant Cartan matrix has vanishing determinant), which has three properties: (i) it is regularly embedded in  $E_{10}$ , (ii) it is generated by electric roots only, and (iii) every node  $P$  in its Dynkin diagram  $\mathbb{D}_{\bar{\mathfrak{g}}}$  is linked to a number  $k$  of nodes that is independent of  $P$  (but depend on the algebra). The Dynkin diagram  $\mathbb{D}_{\bar{\mathfrak{g}}}$  of  $\bar{\mathfrak{g}}$  is actually the line incidence diagram of the geometric configuration  $(n_m, g_3)$  in the sense that (i) each line of  $(n_m, g_3)$  defines a node of  $\mathbb{D}_{\bar{\mathfrak{g}}}$ , and (ii) two nodes of  $\mathbb{D}_{\bar{\mathfrak{g}}}$  are connected by a single bond iff the corresponding lines of  $(n_m, g_3)$  have no point in common. None of the algebras  $\bar{\mathfrak{g}}$  relevant to the truncated models turn out to be hyperbolic: they can be finite, affine, or Lorentzian with infinite-volume Weyl chamber. Because of this, the solutions are non chaotic. After a finite number of collisions, they settle asymptotically into a definite Kasner regime (both in the future and in the past). Disappearance of chaos for diagonal models was also recently observed in [14].

In the most interesting cases,  $\bar{\mathfrak{g}}$  is a rank-10, Lorentzian (but not hyperbolic) Kac-Moody subalgebra of  $E_{10}$ . We do, in fact, get six rank-10 Lorentzian Kac-Moody subalgebras of  $E_{10}$ , which, to our knowledge, have not been previously discussed. We also get one rank-10 Kac-Moody algebra with a Cartan matrix that is degenerate but not positive semi-definite, and hence is not of affine type. Its embedding in  $E_{10}$  involves the quotient by its center. We believe that the display of these subalgebras might be in itself of some mathematical interest in understanding better the structure of  $E_{10}$ . At the level of the corresponding reflection groups, our method exhibits seven rank-10 Coxeter subgroups of

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<sup>2</sup>In the infinite-dimensional case of  $E_{10}$ , the connection between the Lie algebra and the corresponding group is somewhat subtle. We shall proceed formally here, as in the finite-dimensional case. This is possible because, as a rule, the quantities of direct interest for our analysis will be elements of the algebra.

the Weyl group of  $E_{10}$  which have the property that their Coxeter exponents are either 2 or 3 - but never  $\infty$ , and which are furthermore such that there are exactly three edges that meet at each node of their Coxeter diagrams.

Our paper is organized as follows. In the next section, we recall the equations of motion of eleven-dimensional supergravity for time dependent fields (Bianchi type I models) and the consistent truncations associated with geometric configurations. We then recall in section 3 the  $\sigma$ -model formulation of the bosonic sector of eleven-dimensional supergravity. In section 4 we consider regular subalgebras and consistent subgroup truncations of  $\sigma$ -models. This is used in section 5 to relate the consistent truncations of eleven-dimensional supergravity associated with geometric configurations to the consistent truncations of the  $E_{10}$ -sigma model based on regular subalgebras with definite properties that are also spelled out. The method is illustrated in the case of configurations associated with subalgebras of  $E_8$ . We then turn in section 6 to the geometric configurations leading to affine subalgebras, naturally embedded in  $E_9$ . Section 7 is devoted to the rank 10 case. Finally, we close our paper with conclusions and directions for future developments.

## 2. Bianchi I models and eleven-dimensional supergravity

### 2.1 Equations of motion

For time-dependent fields,

$$ds^2 = -N^2(x^0)(dx^0)^2 + G_{ab}(x^0)dx^a dx^b \quad (2.1)$$

$$F_{\lambda\rho\sigma\tau} = F_{\lambda\rho\sigma\tau}(x^0), \quad (2.2)$$

the equations of motion of eleven-dimensional supergravity read

$$\frac{d\left(K^a_b \sqrt{G}\right)}{dx^0} = -\frac{N}{2}\sqrt{G}F^{\alpha\rho\sigma\tau}F_{b\rho\sigma\tau} + \frac{N}{144}\sqrt{G}F^{\lambda\rho\sigma\tau}F_{\lambda\rho\sigma\tau}\delta_b^a \quad (2.3)$$

$$\frac{d\left(F^{0abc}N\sqrt{G}\right)}{dx^0} = \frac{1}{144}\varepsilon^{0abcd_1d_2d_3e_1e_2e_3e_4}F_{0d_1d_2d_3}F_{e_1e_2e_3e_4} \quad (2.4)$$

$$\frac{dF_{a_1a_2a_3a_4}}{dx^0} = 0 \quad (2.5)$$

(dynamical equations) and

$$K^a_b K^b_a - K^2 + \frac{1}{12}F_{\perp abc}F_{\perp}{}^{abc} + \frac{1}{48}F_{abcd}F^{abcd} = 0 \quad (2.6)$$

$$\frac{1}{6}NF^{0bcd}F_{abcd} = 0 \quad (2.7)$$

$$\varepsilon^{0abc_1c_2c_3c_4d_1d_2d_3d_4}F_{c_1c_2c_3c_4}F_{d_1d_2d_3d_4} = 0 \quad (2.8)$$

(Hamiltonian constraint, momentum constraint and Gauss law). Here, we have set  $K_{ab} = (-1/2N)\dot{G}_{ab}$  and  $F_{\perp abc} = (1/N)F_{0abc}$ .

## 2.2 Diagonal metrics and geometric configurations

If the metric is diagonal, the extrinsic curvature  $K_{ab}$  is also diagonal. This is consistent with eq. (2.3) only if  $F^{a\rho\sigma\tau}F_{b\rho\sigma\tau}$  is diagonal, i.e., taking also into account eq. (2.7), if the energy-momentum tensor  $T^\alpha_\beta$  of the 4-form  $F_{\lambda\rho\sigma\tau}$  is diagonal. Assuming zero magnetic field (this restriction will be lifted below), one way to achieve this condition is to assume that the non-vanishing components of the electric field  $F^{\perp abc}$  are determined by “geometric configurations”  $(n_m, g_3)$  with  $n \leq 10$  [1].

A geometric configuration  $(n_m, g_3)$  is a set of  $n$  points and  $g$  lines with the following incidence rules [15–17]:

1. Each line contains three points.
2. Each point is on  $m$  lines.
3. Two points determine at most one line.

It follows that two lines have at most one point in common. It is an easy exercise to verify that  $mn = 3g$ . An interesting question is whether the lines can actually be realized as straight lines in the (real) plane, but, for our purposes, it is not necessary that it should be so; the lines can be bent.

We shall need the configurations with  $n \leq 10$  points. These are all known and are reproduced in the appendix B of [1] and listed in sections 5, 6 and 7 below. There are:

- one configuration  $(3_1, 1_3)$  with 3 points;
- two configurations with 6 points, namely  $(6_1, 2_3)$  and  $(6_2, 4_3)$ ;
- one configuration  $(7_3, 7_3)$ , which is related to the octonions and which cannot be realized by straight lines;
- one configuration  $(8_3, 8_3)$ , which cannot be realized by straight lines;
- one configuration  $(9_1, 3_3)$ , two configurations  $(9_2, 6_3)$ , three configurations  $(9_3, 9_3)$ , and finally one configuration  $(9_4, 12_3)$  that cannot be drawn with straight lines;
- ten configurations  $(10_3, 10_3)$ , with one of them, denoted  $(10_3, 10_3)_1$ , not being realizable in terms of straight lines.

Some of these configurations are related to theorems of projective geometry and are given a name - e.g. the Desargues configuration  $(10_3, 10_3)_3$  explicitly discussed below; but most of them, however, bear no name.

Let  $(n_m, g_3)$  be a geometric configuration with  $n \leq 10$  points. We number the points of the configuration  $1, \dots, n$ . We associate to this geometric configuration a pattern of electric field components  $F^{\perp abc}$  with the following property:  $F^{\perp abc}$  can be non-zero only if the triplet  $(a, b, c)$  is a line of the geometric configuration. If it is not, we take  $F^{\perp abc} = 0$ . It is clear that this property is preserved in time by the equations of motion (in the absence of

magnetic field). Furthermore, because of Rule **3** above, the products  $F^{\perp abc} F^{\perp a' b' c'} g_{bb'} g_{cc'}$  vanish when  $a \neq a'$  so that the energy-momentum tensor is diagonal.

We shall now show that these configurations have an algebraic interpretation in terms of subalgebras of  $E_{10}$ . This will also enable us to relax the condition that the magnetic field should be zero while preserving diagonality. To that end, we need first to recall the  $\sigma$ -model reformulation of eleven-dimensional supergravity.

### 3. Geodesics on the symmetric space $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$ : an overview

#### 3.1 Borel gauge

Let  $\mathfrak{g}$  be the split real form of a rank  $n$  Kac-Moody algebra with generators  $h_i, e_i, f_i$  ( $i = 1, 2, \dots, n$ ) and Cartan matrix  $A_{ij}$  (see [18] for more information). This algebra may be finite or infinite dimensional. We assume in this latter case that the Cartan matrix is symmetrizable and that the symmetrization of  $A_{ij}$  is invertible. In fact, only Lorentzian Kac-Moody algebras, for which the symmetrization of  $A_{ij}$  has signature  $(-, +, +, \dots, +)$ , will be of immediate concern in this paper. We write the triangular decomposition of  $\mathfrak{g}$  as  $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ , where  $\mathfrak{h}$  is the Cartan subalgebra of  $\mathfrak{g}$  and  $\mathfrak{n}_+$  (respectively,  $\mathfrak{n}_-$ ) its upper (respectively, lower) triangular subalgebra containing the “raising” (respectively, “lowering”) operators  $e_i, [e_i, e_j], [e_i, [e_j, e_k]]$  etc (respectively,  $f_i, [f_i, f_j], [f_i, [f_j, f_k]]$  etc) that do not vanish on account of the Serre relations or the Jacobi identity. The raising and lowering operators are collectively called “step operators”.

The split real form of  $\mathfrak{g}$  is the real algebra containing all the real linear combinations of the generators  $h_i, e_i, f_i$  and their multiple commutators. Although non split real forms of Kac-Moody algebras are relevant to some supergravity models [19], we shall for definiteness not consider them explicitly here as the case under central consideration in this paper is  $E_{10,10}$ , the split real form of  $E_{10}$ . In the non split case, it is the real roots and the real Weyl groups that play the role of the roots and Weyl groups introduced below.

Let  $\mathfrak{k}$  be the maximal “compact subalgebra” of  $\mathfrak{g}$ , i.e. the subalgebra pointwise invariant under the Chevalley involution  $\tau$  defined by

$$\tau(h_i) = -h_i, \quad \tau(e_i) = -f_i, \quad \tau(f_i) = -e_i. \tag{3.1}$$

Consider the symmetric space  $\mathcal{G}/\mathcal{K}$ , where  $\mathcal{G}$  is the group obtained by exponentiation of  $\mathfrak{g}$  and  $\mathcal{K}$  is its maximal compact subgroup, obtained by exponentiation of  $\mathfrak{k}$  (as already mentioned previously, in the case where the Cartan matrix is of indefinite type, this is somewhat formal).

The construction of the Lagrangian for the geodesic motion on  $\mathcal{G}/\mathcal{K}$  follows a standard pattern [20, 21]. The motion is formulated in terms of a one-parameter dependent group element  $g(x^0) \in \mathcal{G}$ , with the identification of  $g$  with  $kg$ , where  $k \in \mathcal{K}$ . The Lie algebra element  $\nu(x^0) = \dot{g}g^{-1} \in \mathfrak{g}$  is invariant under multiplication of  $g(x^0)$  to the right by an arbitrary constant group element  $h$ ,  $g(x^0) \rightarrow g(x^0)h$ . Decompose  $\nu(x^0)$  into a part along  $\mathfrak{k}$  and a part perpendicular to  $\mathfrak{k}$ ,

$$\nu(x^0) = Q(x^0) + P(x^0), \quad Q(x^0) = \frac{1}{2}(\nu + \tau(\nu)) \in \mathfrak{k}, \quad P(x^0) = \frac{1}{2}(\nu - \tau(\nu)). \tag{3.2}$$

Under left multiplication by an element  $k(x^0) \in \mathcal{K}$ ,  $Q(x^0)$  behaves as a  $\mathfrak{k}$ -connection, while  $P(x^0)$  transforms in a definite  $\mathfrak{k}$ -representation (which depends on the coset space at hand). The Lagrangian from which the geodesic equations of motion derive is an invariant built out of  $P$  which reads explicitly

$$\mathcal{L} = n(x^0)^{-1} \langle P(x^0) | P(x^0) \rangle \tag{3.3}$$

where  $\langle | \rangle$  is the invariant bilinear form on  $\mathfrak{g}$ . In the case of relevance to gravity, diffeomorphism invariance with respect to time is enforced by introducing the (rescaled) lapse variable  $n(x^0)$ , as done above, whose variation implies that the geodesic on  $\mathcal{G}/\mathcal{K}$  – whose metric has Lorentzian signature – is lightlike. The connection between  $n(x^0)$  and the standard lapse  $N(x^0)$  will be given below.

The Lagrangian is invariant under the gauge transformation  $g(x^0) \rightarrow k(x^0)g(x^0)$  with  $k(x^0) \in \mathcal{K}$ . One can use this gauge freedom to go to the Borel gauge. The Iwasawa decomposition states that  $g$  can be uniquely written as  $g = kan$ , where  $k \in \mathcal{K}$ ,  $a \in \mathcal{H}$  and  $n \in \mathcal{N}_+$ . Here,  $\mathcal{H}$  is the abelian group obtained by exponentiating the Cartan subalgebra  $\mathfrak{h}$  while  $\mathcal{N}_+$  is the group obtained by exponentiating  $\mathfrak{n}_+$ . The Borel gauge is defined by  $k = e$  so that  $g = an$  contains only the Cartan fields (“scale factors”) and the off-diagonal fields associated with the raising operators  $e_\alpha = \{e_i, [e_i, e_j], \dots\}$ , namely,  $a = \exp \beta^i(x^0) h_i$  and  $n = \exp a^\alpha(x^0) e_\alpha$ .

We shall write more explicitly the Lagrangian in the Borel gauge in the  $E_{10}$  case. To that end, we recall first the structure of  $E_{10}$  at low levels.

### 3.2 $E_{10}$ at low levels

We describe the algebra  $E_{10}$  using the level decomposition of [11]. The level zero elements are all the elements of the  $A_{9,9} \equiv \mathfrak{sl}(10, \mathbb{R})$  subalgebra corresponding to the Dynkin subdiagram with nodes 1 to 9, together with the tenth Cartan generator corresponding to the exceptional root labeled “10” in figure 1 and given explicitly in eq. (3.10) below. Thus, the level zero subalgebra is enlarged from  $\mathfrak{sl}(10, \mathbb{R})$  to  $\mathfrak{gl}(10, \mathbb{R}) = \mathfrak{sl}(10, \mathbb{R}) + \mathfrak{h}_{E_{10}}$  (the sum is not direct) and contains all Cartan generators. The  $\mathfrak{sl}(10, \mathbb{R})$  Chevalley generators are given by

$$e_i = K^{i+1}_i \quad f_i = K^i_{i+1} \quad h_i = K^{i+1}_{i+1} - K^i_i \quad (i = 1, \dots, 9), \tag{3.4}$$

while the commutation relations of the level 0 algebra  $\mathfrak{gl}(10, \mathbb{R})$  read

$$[K^a_b, K^c_d] = \delta^c_b K^a_d - \delta^a_d K^c_b. \tag{3.5}$$

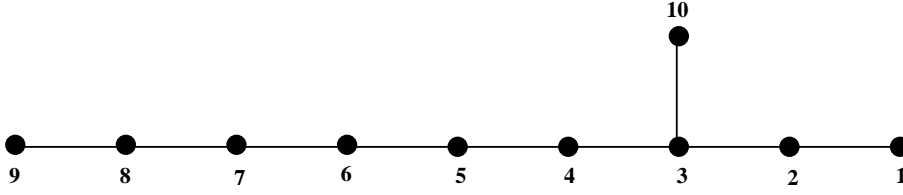
The level 1 generators  $E^{abc}$  and their “transposes”<sup>3</sup>  $F_{abc} = -\tau(E^{abc})$  at level  $-1$  transform contravariantly and covariantly with respect to  $\mathfrak{gl}(10, \mathbb{R})$ ,

$$[K^a_b, E^{cde}] = 3\delta^c_b [E^{de}]^a \quad [K^a_b, F_{cde}] = -3\delta^a_{[c} F_{de]b}. \tag{3.6}$$

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<sup>3</sup>As in [9], we formally define the transpose of a Lie algebra element  $u$  through  $u^T = -\tau(u)$  and extend to formal products - including group elements - through  $(uv)^T = v^T u^T$ . Elements  $k$  of  $\mathcal{K}(E_{10})$  are such that  $k^T = k^{-1}$ . One has also  $Q^T = -Q$  and  $P^T = P$ .





**Figure 1:** The Dynkin diagram of  $E_{10}$ . Labels  $i = 1, \dots, 9$  enumerate the nodes corresponding to simple roots,  $\alpha_i$ , of the  $A_9$  subalgebra and the exceptional node, labeled “10”, is associated to the root  $\alpha_{10}$  that defines the level decomposition.

We further have

$$[E^{abc}, F_{def}] = 18\delta_{[de}^{[ab} K^c]_{f]} - 2\delta_{def}^{abc} \sum_{a=1}^{10} K^a_a, \quad (3.7)$$

where we defined

$$\delta_{cd}^{ab} = \frac{1}{2}(\delta_c^a \delta_d^b - \delta_c^b \delta_d^a), \quad (3.8)$$

and similarly for  $\delta_{def}^{abc}$ . The exceptional generators associated with the roots  $\alpha_{10}$  and  $-\alpha_{10}$  are

$$e_{10} = E^{123} \quad f_{10} = F_{123}. \quad (3.9)$$

From the Chevalley relation  $[e_{10}, f_{10}] = h_{10}$ , one identifies, upon examination of eq. (3.7), the remaining Cartan generator as

$$h_{10} = -\frac{1}{3} \sum_{i \neq 1,2,3} K^a_a + \frac{2}{3}(K^1_1 + K^2_2 + K^3_3). \quad (3.10)$$

It is a straightforward exercise to check from the above commutation relations that the generators  $e_i, e_{10}, f_i, f_{10}, h_i$  and  $h_{10}$  in the  $\mathfrak{gl}(10, \mathbb{R})$ -form given above satisfies indeed the standard Chevalley-Serre relations associated with the Cartan matrix of  $E_{10}$ .

The bilinear form of  $E_{10}$  is given up to level  $\pm 1$  by

$$\langle K^a_b | K^c_d \rangle = \delta_d^a \delta_b^c - \delta_b^a \delta_d^c \quad (3.11)$$

$$\langle E^{abc} | F_{def} \rangle = 3! \delta_{def}^{abc}, \quad (3.12)$$

where the second relation is normalized such that  $\langle E^{123} | F_{123} \rangle = 1$ .

We will explicitly need the generators of  $E_{10}$  up to level 3. These are constructed from multiple commutators of the level 1-generators, i.e. at level 2 we find a 6-form

$$[E^{a_1 a_2 a_3}, E^{a_4 a_5 a_6}] \equiv E^{a_1 \dots a_6} \quad (3.13)$$

(and similarly for the transposes at level  $-2$ ). The commutators at this level are

$$[E^{a_1 \dots a_6}, F_{b_1 \dots b_6}] = 6 \cdot 6! \delta_{[b_1 \dots b_5}^{[a_1 \dots a_5} K^{a_6]_{b_6]} - \frac{2}{3} \cdot 6! \delta_{b_1 \dots b_6}^{a_1 \dots a_6} \sum_{a=1}^{10} K^a_a. \quad (3.14)$$

At level 3 the Jacobi-identity leaves as sole representation occurring in  $E_{10}$  the mixed representation

$$[E^{a_1 a_2 a_3}, E^{a_4 \dots a_9}] = E^{[a_1 | a_2 a_3] \dots a_9}, \quad (3.15)$$

where the level 3-generator  $E^{a_1 | a_2 \dots a_9}$  is antisymmetric in the indices  $a_2 \dots a_9$  and such that antisymmetrizing over all indices gives identically zero,

$$E^{[a_1 | a_2 \dots a_9]} = 0. \quad (3.16)$$

These are all the relations that we will need in this paper. For more details on the decomposition of  $E_{10}$  into representations of  $\mathfrak{gl}(10, \mathbb{R})$  see [22–24].

### 3.3 Lagrangian and conserved currents

The Lagrangian eq. (3.3) has been explicitly written down in the Borel gauge in [11]. Parametrizing the group element  $g(x^0)$  as

$$g(x^0) = \exp X_h(x^0) \exp X_A(x^0) \quad (3.17)$$

where

$$X_h(x^0) = h^a{}_b(x^0) K^b{}_a$$

( $a \geq b$ ) contains the level zero fields and

$$X_A(x^0) = \frac{1}{3!} A_{a_1 a_2 a_3}(x^0) E^{a_1 a_2 a_3} + \frac{1}{6!} A_{a_1 \dots a_6}(x^0) E^{a_1 \dots a_6} + \frac{1}{9!} A_{a_1 | a_2 \dots a_9}(x^0) E^{a_1 | a_2 \dots a_9} + \dots \quad (3.18)$$

contains all fields at positive levels, one finds

$$\begin{aligned} n \mathcal{L} = & \frac{1}{4} (g^{ac} g^{bd} - g^{ad} g^{bc}) \dot{g}_{ab} \dot{g}_{cd} + \frac{1}{2} \frac{1}{3!} DA_{a_1 a_2 a_3} DA^{a_1 a_2 a_3} \\ & + \frac{1}{2} \frac{1}{6!} DA_{a_1 \dots a_6} DA^{a_1 \dots a_6} + \frac{1}{2} \frac{1}{9!} DA_{a_1 | a_2 \dots a_9} DA^{a_1 | a_2 \dots a_9} + \dots \end{aligned} \quad (3.19)$$

Here, the metric  $g_{ab}$  and its inverse are constructed from the level zero vielbein, while the “covariant time derivatives”  $DA$  are defined by

$$\begin{aligned} DA_{a_1 a_2 a_3} &= \dot{A}_{a_1 a_2 a_3} \\ DA_{a_1 \dots a_6} &= \dot{A}_{a_1 \dots a_6} + 10 A_{[a_1 a_2 a_3} \dot{A}_{a_4 a_5 a_6]} \\ DA_{a_1 | a_2 \dots a_9} &= \dot{A}_{a_1 | a_2 \dots a_9} + 42 A_{\langle a_1 a_2 a_3} \dot{A}_{a_4 \dots a_9 \rangle} - 42 \dot{A}_{\langle a_1 a_2 a_3} A_{a_4 \dots a_9 \rangle} \\ &\quad + 280 A_{\langle a_1 a_2 a_3} A_{a_4 a_5 a_6} \dot{A}_{a_7 a_8 a_9 \rangle} \\ &\dots \end{aligned}$$

where  $\langle \rangle$  denotes projection on the level 3 representation. At level  $k$ , each term in  $DA_{a_1 \dots a_{3k}}$  contains one time derivative and is such that the levels match (a typical term in  $DA^{(k)}$  has thus the form  $\dot{A}^{(i_1)} A^{(i_2)} \dots A^{(i_f)}$  with  $i_1 + i_2 + \dots + i_f = k$ ).

The Lagrangian is not only gauge invariant under left multiplication by an arbitrary time-dependent element of  $\mathcal{K}(\mathcal{E}_{10})$ , it is also invariant under right multiplication by an

arbitrary constant element of  $\mathcal{E}_{10}$ . Invariance under this rigid symmetry leads to an infinite set of  $E_{10}$ -valued conserved currents [9, 11], which are, in the gauge  $n = 1$ ,

$$J = g^{-1}Pg = \frac{1}{2}\mathcal{M}^{-1}\dot{\mathcal{M}} \tag{3.20}$$

where the gauge invariant infinite “symmetric” matrix  $\mathcal{M}$  is defined by

$$\mathcal{M} = g^T g. \tag{3.21}$$

The current fulfills

$$J^T \mathcal{M} = \mathcal{M}J. \tag{3.22}$$

Eq. (3.20) can formally be integrated to yield

$$\mathcal{M}(x^0) = \mathcal{M}(0)e^{2x^0 J} = e^{x^0 J^T} \mathcal{M}(0)e^{x^0 J}. \tag{3.23}$$

From this, one can read off the group element,

$$g(x^0) = k(x^0)g(0)e^{x^0 J} \tag{3.24}$$

where the compensating  $\mathcal{K}(\mathcal{E}_{10})$ -transformation  $k(x^0)$  is such that  $g(x^0)$  remains in the Borel gauge. The explicit determination of  $k(x^0)$  may be quite a hard task.

### 3.4 Consistent truncations

The  $\sigma$ -model can be truncated in various consistent ways. By “consistent truncation”, we mean a truncation to a sub-model whose solutions are also solutions of the full model.

#### 3.4.1 Level truncation

The first useful truncation was discussed in [9, 11] and consists in setting all covariant derivatives of the fields above a given level equal to zero. This is equivalent to equating to zero the momenta conjugate to the  $\sigma$ -model variables above that given level.

Imposing in particular  $DA^{(k)} = 0$  for  $k \geq 3$  leads to equations of motion which are not only consistent from the  $\sigma$ -model point of view, but which are also equivalent to the dynamical equations of motion of eleven-dimensional supergravity restricted to homogeneous fields  $G_{ab}(t)$  and  $F_{\alpha\beta\gamma\delta}(t)$  (and no fermions). The dictionary that makes the equivalence between the  $\sigma$ -model and supergravity is given by [11]

$$g_{ab} = G_{ab} \tag{3.25}$$

$$DA_{a_1 a_2 a_3} = F_{0a_1 a_2 a_3} \tag{3.26}$$

$$DA^{a_1 a_2 a_3 a_4 a_5 a_6} = -\frac{n}{4!} \epsilon^{a_1 a_2 a_3 a_4 a_5 a_6 b_1 b_2 b_3 b_4} F_{b_1 b_2 b_3 b_4} \tag{3.27}$$

together with  $n = N/\sqrt{G}$ . Furthermore, the  $\sigma$ -model constraint obtained by varying  $n(x^0)$ , which enforces reparametrization invariance, is just the supergravity Hamiltonian constraint (2.6) in the homogeneous setting.

Full equivalence of the level 3 truncated  $\sigma$ -model with spatially homogeneous supergravity requires that one imposes also the momentum constraint as well as the Gauss law.

### 3.4.2 Subgroup truncation

Another way to consistently truncate the  $\sigma$ -model equations of motion is to restrict the dynamics to an appropriately chosen subgroup.

We shall consider here only subgroups obtained by exponentiating regular subalgebras of  $\mathfrak{g}$ , a concept to which we now turn.

## 4. Regular subalgebras

### 4.1 Definitions

Let  $\bar{\mathfrak{g}}$  be a Kac-Moody subalgebra of  $\mathfrak{g}$ , with triangular decomposition  $\bar{\mathfrak{g}} = \bar{\mathfrak{n}}_- \oplus \bar{\mathfrak{h}} \oplus \bar{\mathfrak{n}}_+$ . We assume that  $\bar{\mathfrak{g}}$  is canonically embedded in  $\mathfrak{g}$ , i.e., that the Cartan subalgebra  $\bar{\mathfrak{h}}$  of  $\bar{\mathfrak{g}}$  is a subalgebra of the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$ ,  $\bar{\mathfrak{h}} \subset \mathfrak{h}$ , so that  $\bar{\mathfrak{h}} = \bar{\mathfrak{g}} \cap \mathfrak{h}$ . We shall say that  $\bar{\mathfrak{g}}$  is regularly embedded in  $\mathfrak{g}$  (and call it a “regular subalgebra”) iff two conditions are fulfilled : (i) the step operators of  $\bar{\mathfrak{g}}$  are step operators of  $\mathfrak{g}$ ; and (ii) the simple roots of  $\bar{\mathfrak{g}}$  are real roots of  $\mathfrak{g}$ . It follows that the Weyl group of  $\bar{\mathfrak{g}}$  is a subgroup of the Weyl group of  $\mathfrak{g}$  and that the root lattice of  $\bar{\mathfrak{g}}$  is a sublattice of the root lattice of  $\mathfrak{g}$ .

The second condition is automatic in the finite-dimensional case where there are only real roots. It must be separately imposed in the general case. Consider for instance the rank 2 Kac-Moody algebra  $\mathcal{A}$  with Cartan matrix

$$\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}.$$

Let

$$x = \frac{1}{\sqrt{3}}[e_1, e_2] \tag{4.1}$$

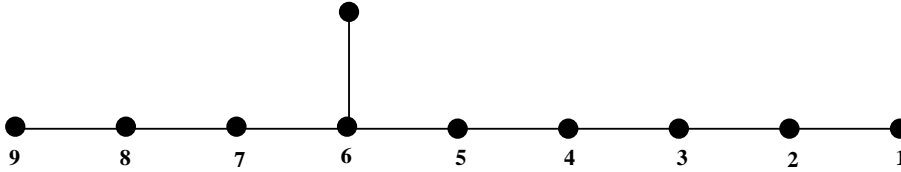
$$y = \frac{1}{\sqrt{3}}[f_1, f_2] \tag{4.2}$$

$$z = -(h_1 + h_2). \tag{4.3}$$

It is easy to verify that  $x, y, z$  define an  $A_1$  subalgebra of  $\mathcal{A}$  since  $[z, x] = 2x$ ,  $[z, y] = -2y$  and  $[x, y] = z$ . Moreover, the Cartan subalgebra of  $A_1$  is a subalgebra of the Cartan subalgebra of  $\mathcal{A}$ , and the step operators of  $A_1$  are step operators of  $\mathcal{A}$ . However, the simple root  $\alpha = \alpha_1 + \alpha_2$  of  $A_1$  (which is an  $A_1$ -real root since  $A_1$  is finite-dimensional), is an imaginary root of  $\mathcal{A}$ :  $\alpha_1 + \alpha_2$  has norm squared equal to  $-2$ . Even though the root lattice of  $A_1$  (namely,  $\{\pm\alpha\}$ ) is a sublattice of the root lattice of  $\mathcal{A}$ , the reflection in  $\alpha$  is not a Weyl reflection of  $\mathcal{A}$ . According to our definition, this embedding of  $A_1$  in  $\mathcal{A}$  is not a regular embedding.

### 4.2 Examples

We shall be interested in regular subalgebras of  $E_{10}$ .



**Figure 2:** The Dynkin diagram of  $\mathcal{B} \equiv E_7^{+++}$ . The root without number is the root denoted  $\bar{\alpha}_{10}$  in the text.

#### 4.2.1 $A_9 \subset \mathcal{B} \subset E_{10}$

A first, simple, example of a regular embedding is the embedding of  $A_9$  in  $E_{10}$  used to define the level. This is not a maximal embedding since one can find a proper subalgebra  $\mathcal{B}$  of  $E_{10}$  that contains  $A_9$ . One may take for  $\mathcal{B}$  the Kac-Moody subalgebra of  $E_{10}$  generated by the operators at levels 0 and  $\pm 2$ , which is a subalgebra of the algebra containing all operators of even level<sup>4</sup>. It is regularly embedded in  $E_{10}$ . Its Dynkin diagram is shown on figure 2.

In terms of the simple roots of  $E_{10}$ , the simple roots of  $\mathcal{B}$  are  $\alpha_1$  through  $\alpha_9$  and  $\bar{\alpha}_{10} = 2\alpha_{10} + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$ . The algebra  $\mathcal{B}$  is Lorentzian but not hyperbolic. It can be identified with the “very extended” algebra  $E_7^{+++}$  [25].

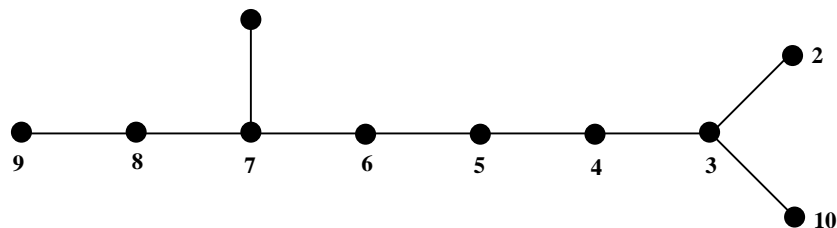
#### 4.2.2 $DE_{10} \subset E_{10}$

In [26], Dynkin has given a method for finding all maximal regular subalgebras of finite-dimensional simple Lie algebras. The method is based on using the highest root and is not generalizable as such to general Kac-Moody algebras for which there is no highest root. Nevertheless, it is useful for constructing regular embeddings of overextensions of finite dimensional simple Lie algebras. We illustrate this point in the case of  $E_8$  and its overextension  $E_{10} \equiv E_8^{+++}$ . In the notation of figure 1, the simple roots of  $E_8$  (which is regularly embedded in  $E_{10}$ ) are  $\alpha_1, \dots, \alpha_7$  and  $\alpha_{10}$ .

Applying Dynkin’s procedure to  $E_8$ , one easily finds that  $D_8$  can be regularly embedded in  $E_8$ . The simple roots of  $D_8 \subset E_8$  are  $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{10}$  and  $\beta \equiv -\theta$ , where  $\theta = 3\alpha_{10} + 6\alpha_3 + 4\alpha_2 + 2\alpha_1 + 5\alpha_4 + 4\alpha_5 + 3\alpha_6 + 2\alpha_7$  is the highest root of  $E_8$  (which, incidentally, has height 29). One can replace this embedding, in which a simple root of  $D_8$ , namely  $\beta$ , is a negative root of  $E_8$  (and the corresponding raising operator of  $D_8$  is a lowering operator for  $E_8$ ), by an equivalent one in which all simple roots of  $D_8$  are positive roots of  $E_8$ .

This is done as follows. It is reasonable to guess that the searched-for Weyl element that maps the “old”  $D_8$  on the “new”  $D_8$  is some product of the Weyl reflections in the four  $E_8$ -roots orthogonal to the simple roots  $\alpha_3, \alpha_4, \alpha_5, \alpha_6$  and  $\alpha_7$ , expected to be shared (as simple roots) by  $E_8$ , the old  $D_8$  and the new  $D_8$  - and therefore to be invariant under the searched-for Weyl element. This guess turns out to be correct: under the action of the product of the commuting  $E_8$ -Weyl reflections in the  $E_8$ -roots  $\mu_1 = 2\alpha_1 + 3\alpha_2 + 5\alpha_3 +$

<sup>4</sup>We thank Axel Kleinschmidt for an informative comment on this point.



**Figure 3:**  $DE_{10} \equiv D_8^{++}$  regularly embedded in  $E_{10}$ . Labels  $2, \dots, 10$  represent the simple roots  $\alpha_2, \dots, \alpha_{10}$  of  $E_{10}$  and the unlabeled node corresponds to the positive root  $\bar{\beta} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_{10}$ .

$4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 3\alpha_{10}$  and  $\mu_2 = 2\alpha_1 + 4\alpha_2 + 5\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7 + 2\alpha_{10}$ , the set of  $D_8$ -roots  $\{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_{10}, \beta\}$  is mapped on the equivalent set of positive roots  $\{\alpha_{10}, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_2, \bar{\beta}\}$  where  $\bar{\beta} = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 + 2\alpha_{10}$ . In this equivalent embedding, all raising operators of  $D_8$  are also raising operators of  $E_8$ . What is more, the highest root of  $D_8$ ,  $\theta_{D_8} = \alpha_{10} + 2\alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_2 + \bar{\beta}$  is equal to the highest root of  $E_8$ . Because of this, the affine root  $\alpha_8$  of the untwisted affine extension  $E_8^+$  can be identified with the affine root of  $D_8^+$ , and the overextended root  $\alpha_9$  can also be taken to be the same. Hence,  $DE_{10}$  can be regularly embedded in  $E_{10}$  (see figure 3).

The embedding just described is in fact relevant to string theory and has been discussed from various points of view in previous papers [27, 28]. By dimensional reduction of the bosonic sector of eleven-dimensional supergravity on a circle, one gets, after dropping the Kaluza-Klein vector and the 3-form, the bosonic sector of pure N=1 ten-dimensional supergravity. The simple roots of  $DE_{10}$  are the symmetry walls and the electric and magnetic walls of the 2-form and coincide with the positive roots given above [3].

A similar construction shows that  $A_8^{++}$  can be regularly embedded in  $E_{10}$ , and that  $DE_{10}$  can be regularly embedded in  $BE_{10} \equiv B_8^{++}$ .

### 4.3 Further properties

As we have just seen, the raising operators of  $\bar{\mathfrak{g}}$  might be raising or lowering operators of  $\mathfrak{g}$ . We shall consider here only the case when the positive (respectively, negative) step operators of  $\bar{\mathfrak{g}}$  are also positive (respectively, negative) step operators of  $\mathfrak{g}$ , so that  $\bar{\mathfrak{n}}_- = \mathfrak{n}_- \cap \bar{\mathfrak{g}}$  and  $\bar{\mathfrak{n}}_+ = \mathfrak{n}_+ \cap \bar{\mathfrak{g}}$  (“positive regular embeddings”). This will always be assumed from now on.

In the finite dimensional case, there is a useful criterion to determine regular algebras from subsets of roots. This criterion has been generalized to Kac-Moody algebras in [29]. It goes as follows.

**Theorem:** Let  $\Phi_{real}^+$  be the set of positive real roots of a Kac-Moody algebra  $\mathcal{A}$ . Let  $\beta_1, \dots, \beta_n \in \Phi_{real}^+$  be chosen such that none of the differences  $\beta_i - \beta_j$  is a root of  $\mathcal{A}$ . Assume furthermore that the  $\beta_i$ ’s are such that the matrix  $C = [C_{ij}] = [2\langle \beta_i | \beta_j \rangle / \langle \beta_i | \beta_i \rangle]$  has non-vanishing determinant. For each  $1 \leq i \leq n$ , choose non-zero root vectors  $E_i$  and  $F_i$  in the one-dimensional root spaces corresponding to the positive real roots  $\beta_i$  and the

negative real roots  $-\beta_i$ , respectively, and let  $H_i = [E_i, F_i]$  be the corresponding element in the Cartan subalgebra of  $\mathcal{A}$ . Then, the (regular) subalgebra of  $\mathcal{A}$  generated by  $\{E_i, F_i, H_i\}$ ,  $i = 1, \dots, n$ , is a Kac-Moody algebra with Cartan matrix  $[C_{ij}]$ .

**Proof:** The proof of this theorem is given in [29]. Note that the Cartan integers  $2 \frac{\langle \beta_i | \beta_j \rangle}{\langle \beta_i | \beta_i \rangle}$  are indeed integers (because the  $\beta_i$ 's are positive real roots), which are non positive (because  $\beta_i - \beta_j$  is not a root), so that  $[C_{ij}]$  is a Cartan matrix.

**Comments:**

1. When the Cartan matrix is degenerate, the corresponding Kac-Moody algebra has non trivial ideals [18]. Verifying that the Chevalley-Serre relations are fulfilled is not sufficient to guarantee that one gets the Kac-Moody algebra corresponding to the Cartan matrix  $[C_{ij}]$  since there might be non trivial quotients. We will in fact precisely encounter below situations in which the algebra generated by the set  $\{E_i, F_i, H_i\}$  is the quotient of the Kac-Moody algebra with Cartan matrix  $[C_{ij}]$  by a non trivial ideal.
2. If the matrix  $[C_{ij}]$  is decomposable, say  $C = D \oplus E$  with  $D$  and  $E$  indecomposable, then the Kac-Moody algebra  $\mathbb{KM}(C)$  generated by  $C$  is the direct sum of the Kac-Moody algebra  $\mathbb{KM}(D)$  generated by  $D$  and the Kac-Moody algebra  $\mathbb{KM}(E)$  generated by  $E$ . The subalgebras  $\mathbb{KM}(D)$  and  $\mathbb{KM}(E)$  are ideals. If  $C$  has non-vanishing determinant, then both  $D$  and  $E$  have non-vanishing determinant. Accordingly,  $\mathbb{KM}(D)$  and  $\mathbb{KM}(E)$  are simple [18] and hence, either occur faithfully or trivially. Because the generators  $E_i$  are linearly independent, both  $\mathbb{KM}(D)$  and  $\mathbb{KM}(E)$  occur faithfully. Therefore, in the above theorem the only case that requires special treatment is when the Cartan matrix  $C$  has vanishing determinant.

It is convenient to universally normalize the Killing form of Kac-Moody algebras in such a way that the long real roots have always the same squared length, conveniently taken equal to two. It is then easily seen that the Killing form of any regular Kac-Moody subalgebra of  $E_{10}$  coincides with the invariant form induced from the Killing form of  $E_{10}$  through the embedding (“Dynkin index equal to one”) since  $E_{10}$  is “simply laced”. This property does not hold for non regular embeddings as the example given in subsection 4.1, which has Dynkin index  $-1$ , shows.

**4.4 Reductive subalgebras**

We shall also consider embeddings in a Kac-Moody algebra  $\mathcal{A}$  of algebras  $\mathcal{B} = \mathcal{D} \oplus \mathbb{R}^k$  which are the direct sums of a Kac-Moody algebra  $\mathcal{D}$  plus an Abelian algebra  $\mathbb{R}^k$ . One says that the embedding is regular if  $\mathcal{D}$  is regularly embedded in the above sense and if  $\mathbb{R}^k$  is a subalgebra of the Cartan subalgebra  $H_{\mathcal{A}}$  of  $\mathcal{A}$ . The abelian algebra  $H_{\mathcal{D}} \oplus \mathbb{R}^k$  is called the Cartan subalgebra of  $\mathcal{B}$ . We take for invariant bilinear form on  $\mathbb{R}^k$  the invariant form induced from the Killing form of  $\mathcal{A}$  through the embedding.

#### 4.5 Back to subgroup truncations

We now come back to consistent subgroup truncations of the non-linear sigma model  $\mathcal{G}/\mathcal{K}$ .

Let  $\bar{\mathcal{G}}$  be the subgroup of  $\mathcal{G}$  obtained by exponentiating a regular subalgebra  $\bar{\mathfrak{g}}$  of  $\mathfrak{g}$ . Assume that the initial conditions  $g(0)$  and  $\dot{g}(0)$  are such that (i) the group element  $g(0)$  is in  $\bar{\mathcal{G}}$ ; and (ii) the conserved current  $J$  is in  $\bar{\mathfrak{g}}$ . Then,  $g(0)\exp(x^0 J)$  belongs to  $\bar{\mathcal{G}}$  for all  $x^0$ . Furthermore, there exists an element  $k(x^0) \in \bar{\mathcal{K}}$  (the maximal compact subgroup of  $\bar{\mathcal{G}}$ ) such that  $k(x^0)g(0)\exp(x^0 J)$  fulfills the Borel gauge from the point of view of  $\bar{\mathcal{G}}$ . Because the embedding is regular,  $k(x^0)g(0)\exp(x^0 J)$  fulfills also the Borel gauge from the point of view of  $\mathcal{G}$ ;  $k(x^0)$  belongs in fact also to  $\mathcal{K}$ . But  $k(x^0)g(0)\exp(x^0 J)$  is precisely the solution of the equations of motion (see eq. (3.24)). This shows that one can consistently truncate the dynamics of the non-linear sigma model  $\mathcal{G}/\mathcal{K}$  to  $\bar{\mathcal{G}}/\bar{\mathcal{K}}$  since initial conditions in  $\bar{\mathcal{G}}$  remain in  $\bar{\mathcal{G}}$ .

Finally, because the Killing forms coincide, the constraints resulting from time reparametrization invariance also agree.

The consistent truncation to a regular subgroup was used in [30] to investigate the compatibility of the non linear  $\sigma$ -model  $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$  with the non linear  $\sigma$ -model  $\mathcal{E}_{11}/\mathcal{K}(\mathcal{E}_{11})$ . Another interesting consistent truncation is the truncation to the regular subalgebra  $E_8 \oplus A_1 \oplus \mathbb{R}l$ . Here,  $E_8 \oplus A_1$  is obtained by deleting the node numbered 8 in figure 1, and  $\mathbb{R}l$  is the one-dimensional subalgebra of the Cartan subalgebra  $\mathfrak{h}_{E_{10}}$  of  $E_{10}$  containing the multiples of  $l$ , where  $l$  is orthogonal to both the Cartan subalgebra  $\mathfrak{h}_{E_8}$  of  $E_8$  and the Cartan subalgebra  $\mathfrak{h}_{A_1}$  of  $A_1$ . The subalgebra  $\mathbb{R}l$  is timelike. Explicitly,  $l = K^9_9 + K^{10}_{10}$ . The restriction to  $E_8 \oplus A_1 \oplus \mathbb{R}l$  is not only consistent with the sigma-model equations, but also with the supergravity equations of motion because there is no root in  $E_8 \oplus A_1$  of height  $> 29$  and hence the argument of [11] applies. One way to realize the truncation is to dimensionally reduce eleven-dimensional supergravity on the 3-torus, dualize all fields (except the three-dimensional metric) to scalars, and then impose that the fields (metric and scalar fields) depend only on time. Although non chaotic, this truncation is interesting because it involves some level 3 fields, corresponding to “curvature walls” (the Kaluza-Klein vector components of the eleven-dimensional metric do depend on space if their duals depend only on time, so that there is some spatial curvature). One can further truncate to the subalgebra  $E_8 \oplus \mathbb{R}l$  by assuming that the metric is diagonal with equal diagonal components. This case was thoroughly investigated in [31, 32].

### 5. Geometric configurations and regular subalgebras of $E_{10}$

We will now apply the machinery from the previous sections to reveal a “duality” between the geometric configurations and a class of regular subalgebras of  $E_{10}$ .

#### 5.1 General considerations

In order to match diagonal Bianchi I cosmologies with the  $\sigma$ -model, one must truncate the  $\mathcal{E}_{10}/\mathcal{K}(\mathcal{E}_{10})$  Lagrangian in such a way that the metric  $g_{ab}$  is diagonal. This will be the case if the subalgebra  $S$  to which one truncates has no generator  $K^i_j$  with  $i \neq j$ . Indeed,



the off-diagonal components of the metric are precisely the exponentials of the associated  $\sigma$ -model fields. The set of simple roots of  $S$  should therefore not contain any level zero root.

Consider “electric” regular subalgebras of  $E_{10}$ , for which the simple roots are all at level one, where the 3-form electric field variables live. These roots can be parametrized by 3 indices corresponding to the indices of the electric field, with  $i_1 < i_2 < i_3$ . We denote them  $\alpha_{i_1 i_2 i_3}$ . For instance,  $\alpha_{123} \equiv \alpha_{10}$ . In terms of the  $\beta$ -parametrization of [3, 9], one has  $\alpha_{i_1 i_2 i_3} = \beta^{i_1} + \beta^{i_2} + \beta^{i_3}$ .

Now, for  $S$  to be a regular subalgebra, it must fulfill, as we have seen, the condition that the difference between any two of its simple roots is not a root of  $E_{10}$ :  $\alpha_{i_1 i_2 i_3} - \alpha_{i'_1 i'_2 i'_3} \notin \Phi_{E_{10}}$  for any pair  $\alpha_{i_1 i_2 i_3}$  and  $\alpha_{i'_1 i'_2 i'_3}$  of simple roots of  $S$ . But one sees by inspection of the commutator of  $E^{i_1 i_2 i_3}$  with  $F_{i'_1 i'_2 i'_3}$  in eq. (3.7) that  $\alpha_{i_1 i_2 i_3} - \alpha_{i'_1 i'_2 i'_3}$  is a root of  $E_{10}$  if and only if the sets  $\{i_1, i_2, i_3\}$  and  $\{i'_1, i'_2, i'_3\}$  have exactly two points in common. For instance, if  $i_1 = i'_1$ ,  $i_2 = i'_2$  and  $i_3 \neq i'_3$ , the commutator of  $E^{i_1 i_2 i_3}$  with  $F_{i'_1 i'_2 i'_3}$  produces the off-diagonal generator  $K^{i_3}_{i'_3}$  corresponding to a level zero root of  $E_{10}$ . In order to fulfill the required condition, one must avoid this case, i.e., one must choose the set of simple roots of the electric regular subalgebra  $S$  in such a way that given a pair of indices  $(i_1, i_2)$ , there is at most one  $i_3$  such that the root  $\alpha_{i_j k}$  is a simple root of  $S$ , with  $(i, j, k)$  the re-ordering of  $(i_1, i_2, i_3)$  such that  $i < j < k$ .

To each of the simple roots  $\alpha_{i_1 i_2 i_3}$  of  $S$ , one can associate the line  $(i_1, i_2, i_3)$  connecting the three points  $i_1$ ,  $i_2$  and  $i_3$ . If one does this, one sees that the above condition is equivalent to the following statement: *the set of points and lines associated with the simple roots of  $S$  must fulfill the third Rule defining a geometric configuration, namely, that two points determine at most one line*. Thus, this geometric condition has a nice algebraic interpretation in terms of regular subalgebras of  $E_{10}$ .

The first rule, which states that each line contains 3 points, is a consequence of the fact that the  $E_{10}$ -generators at level one are the components of a 3-index antisymmetric tensor. The second rule, that each point is on  $m$  lines, is less fundamental from the algebraic point of view since it is not required to hold for  $S$  to be a regular subalgebra. It was imposed in [1] in order to allow for solutions isotropic in the directions that support the electric field. We keep it here as it yields interesting structure (see next subsection). We briefly discuss in the conclusions what happens when this condition is lifted.

## 5.2 Incidence diagrams and Dynkin diagrams

We have just shown that each geometric configuration  $(n_m, g_3)$  with  $n \leq 10$  defines a regular subalgebra  $S$  of  $E_{10}$ . In order to determine what this subalgebra  $S$  is, one needs, according to the theorem recalled in section 4, to compute the Cartan matrix

$$C = [C_{i_1 i_2 i_3, i'_1 i'_2 i'_3}] = \left[ \left\langle \alpha_{i_1 i_2 i_3} \mid \alpha_{i'_1 i'_2 i'_3} \right\rangle \right] \tag{5.1}$$

(the real roots of  $E_{10}$  have squared length equal to 2). According to that same theorem, the algebra  $S$  is then just the rank  $g$  Kac-Moody algebra with Cartan matrix  $C$ , unless  $C$  has zero determinant, in which case  $S$  might be the quotient of that algebra by a non trivial ideal.

Using for instance the root parametrization of [3, 9] and the expression of the scalar product in terms of this parametrization, one easily verifies that the scalar product  $\langle \alpha_{i_1 i_2 i_3} | \alpha_{i'_1 i'_2 i'_3} \rangle$  is equal to:

$$\langle \alpha_{i_1 i_2 i_3} | \alpha_{i'_1 i'_2 i'_3} \rangle = 2 \quad \text{if all three indices coincide,} \quad (5.2)$$

$$= 1 \quad \text{if two and only two indices coincide,} \quad (5.3)$$

$$= 0 \quad \text{if one and only one index coincides,} \quad (5.4)$$

$$= -1 \quad \text{if no indices coincide.} \quad (5.5)$$

The second possibility does not arise in our case since we deal with geometric configurations. For completeness, we also list the scalar products of the electric roots  $\alpha_{ijk}$  ( $i < j < k$ ) with the symmetry roots  $\alpha_{\ell m}$  ( $\ell < m$ ) associated with the raising operators  $K_\ell^m$ :

$$\langle \alpha_{ijk} | \alpha_{\ell m} \rangle = -1 \quad \text{if } \ell \in \{i, j, k\} \text{ and } m \notin \{i, j, k\} \quad (5.6)$$

$$= 0 \quad \text{if } \{\ell, m\} \subset \{i, j, k\} \text{ or } \{\ell, m\} \cap \{i, j, k\} = \emptyset, \quad (5.7)$$

$$= 1 \quad \text{if } \ell \notin \{i, j, k\} \text{ and } m \in \{i, j, k\} \quad (5.8)$$

as well as the scalar products of the symmetry roots among themselves,

$$\langle \alpha_{ij} | \alpha_{\ell m} \rangle = -1 \quad \text{if } j = \ell \text{ or } i = m, \quad (5.9)$$

$$= 0 \quad \text{if } \{\ell, m\} \cap \{i, j\} = \emptyset, \quad (5.10)$$

$$= 1 \quad \text{if } i = \ell \text{ or } j \neq m, \quad (5.11)$$

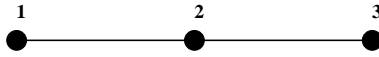
$$= 2 \quad \text{if } \{\ell, m\} = \{i, j\}. \quad (5.12)$$

Given a geometric configuration  $(n_m, g_3)$ , one can associate with it a “line incidence diagram” that encodes the incidence relations between its lines. To each line of  $(n_m, g_3)$  corresponds a node in the incidence diagram. Two nodes are connected by a single bond if and only if they correspond to lines with no common point (“parallel lines”). Otherwise, they are not connected<sup>5</sup>. By inspection of the above scalar products, we come to the important conclusion that *the Dynkin diagram of the regular, rank  $g$ , Kac-Moody subalgebra  $S$  associated with the geometric configuration  $(n_m, g_3)$  is just its line incidence diagram*. We shall call the Kac-Moody algebra  $S$  the algebra “dual” to the geometric configuration  $(n_m, g_3)$ .

Because the geometric configurations have the property that the number of lines through any point is equal to a constant  $m$ , the number of lines parallel to any given line is equal to a number  $k$  that depends only on the configuration and not on the line. This is in fact true in general and not only for  $n \leq 10$  as can be seen from the following

---

<sup>5</sup>One may also consider a point incidence diagram defined as follows: the nodes of the point incidence diagram are the points of the geometric configuration. Two nodes are joined by a single bond if and only if there is no straight line connecting the corresponding points. The point incidence diagrams of the configurations  $(9_3, 9_3)$  are given in [16]. For these configurations, projective duality between lines and points lead to identical line and point incidence diagrams. Unless otherwise stated, the expression “incidence diagram” will mean “line incidence diagram”.



**Figure 4:**  $(3_1, 1_3)$ : The only allowed configuration for  $n = 3$ .

argument. For a configuration with  $n$  points,  $g$  lines and  $m$  lines through each point, any given line  $\Delta$  admits  $3(m - 1)$  true secants, namely,  $(m - 1)$  through each of its points<sup>6</sup>. By definition, these secants are all distinct since none of the lines that  $\Delta$  intersects at one of its points, say  $P$ , can coincide with a line that it intersects at another of its points, say  $P'$ , since the only line joining  $P$  to  $P'$  is  $\Delta$  itself. It follows that the total number of lines that  $\Delta$  intersects is the number of true secants plus  $\Delta$  itself, i.e.  $3(m - 1) + 1$ . As a consequence, each line in the configuration admits  $k = g - [3(m - 1) + 1]$  parallel lines, which is then reflected in the fact that each node in the associated Dynkin diagram has the same number,  $k$ , of adjacent nodes.

### 5.3 Geometric configuration $(3_1, 1_3)$

To illustrate the discussion, we begin by constructing the algebra associated to the simplest configuration  $(3_1, 1_3)$ . This example also exhibits some subtleties associated with the Hamiltonian constraint and the ensuing need to extend  $S$  when the algebra dual to the geometric configuration is finite-dimensional.

In light of our discussion, considering the geometric configuration  $(3_1, 1_3)$  is equivalent to turning on only the component  $A_{123}(x^0)$  of the 3-form that multiplies the generator  $E^{123}$  in the group element  $g$  and the diagonal metric components corresponding to the Cartan generator  $h = [E^{123}, F_{123}]$ . The algebra has thus basis  $\{e, f, h\}$  with

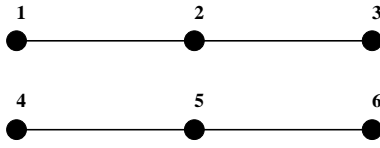
$$e \equiv E^{123} \quad f \equiv F_{123} \quad h = [e, f] = -\frac{1}{3} \sum_{a \neq 1, 2, 3} K^a_a + \frac{2}{3} (K^1_1 + K^2_2 + K^3_3). \quad (5.13)$$

The Cartan matrix is just (2) and is not degenerate. It defines an  $A_1$  regular subalgebra. The Chevalley-Serre relations, which are guaranteed to hold according to the general argument, are easily verified. The configuration  $(3_1, 1_3)$  is thus dual to  $A_1$ .

This  $A_1$  algebra is simply the  $\mathfrak{sl}(2)$ -algebra associated with the simple root  $\alpha_{10}$ . Because the Killing form on  $A_1$  is positive definite, one cannot find a solution of the Hamiltonian constraint if one turns on only  $A_1$ . One needs to enlarge  $A_1$  (at least) by a one-dimensional subalgebra  $\mathbb{R}l$  of  $\mathfrak{h}_{E_{10}}$  that is timelike. One can take for  $l$  the Cartan element  $K^4_4 + K^5_5 + K^6_6 + K^7_7 + K^8_8 + K^9_9 + K^{10}_{10}$ , which ensures isotropy in the directions not supporting the electric field. Thus, the appropriate regular subalgebra of  $E_{10}$  in this case is  $A_1 \oplus \mathbb{R}l$ . This construction reproduces the ‘‘SM2-brane’’ solution given in section 3 of [1], describing two asymptotic Kasner regimes separated by a collision against an electric wall.

The need to enlarge the algebra  $A_1$  was discussed in the paper [33] where a group theoretical interpretation of some cosmological solutions of eleven dimensional supergravity

<sup>6</sup>A true secant is here defined as a line, say  $\Delta'$ , distinct from  $\Delta$  and with a non-empty intersection with  $\Delta$ .



**Figure 5:**  $(6_1, 2_3)$ : The simplest configuration allowed for  $n = 6$ .



**Figure 6:** The Dynkin diagram of  $A_2$ , dual to the configuration  $(6_1, 2_3)$ .

was given. In that paper, it was also observed that  $\mathbb{R}l$  can be viewed as the Cartan subalgebra of the (non regularly embedded) subalgebra  $A_1$  associated with an imaginary root at level 21, but since the corresponding field is not excited, the relevant subalgebra is really  $\mathbb{R}l$ .

#### 5.4 Geometric configuration $(6_1, 2_3)$

For  $n = 6$  we start with the double-line configuration,  $(6_1, 2_3)$ , in figure 5.

This graph yields the generators

$$\begin{aligned}
 e_1 &\equiv E^{123}, & f_1 &\equiv F_{123}, & h_1 &\equiv -\frac{1}{3} \sum_{a \neq 1,2,3} K^a_a + \frac{2}{3}(K^1_1 + K^2_2 + K^3_3) \\
 e_2 &\equiv E^{456}, & f_2 &\equiv F_{456}, & h_2 &\equiv -\frac{1}{3} \sum_{a \neq 4,5,6} K^a_a + \frac{2}{3}(K^4_4 + K^5_5 + K^6_6)
 \end{aligned} \tag{5.14}$$

with the following commutators,

$$[e_i, f_i] = h_i \quad [h_i, e_i] = 2e_i \quad [h_i, f_i] = -2f_i \quad (i = 1, 2). \tag{5.15}$$

Using the rules outlined above, the Cartan matrix is easily found to be

$$A_{(6_1, 2_3)} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, \tag{5.16}$$

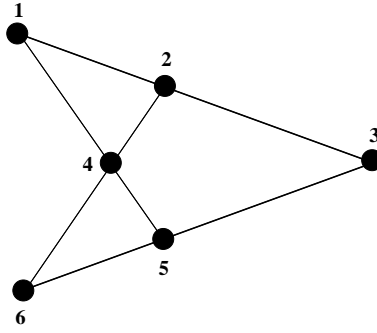
which is the Cartan matrix of  $A_2$ ,  $A_{(6_1, 2_3)} = A_2$ . Thus, the configuration  $(6_1, 2_3)$  in figure 5 is dual to  $A_2$ , whose Dynkin diagram is shown in figure 6.

Note that the roots  $\alpha_{123}$  and  $\alpha_{456}$  are  $\alpha_{10}$  and  $\alpha_{10} + \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5$  so that this subalgebra is in fact already a (non maximal) regular subalgebra of  $E_6$ . The generator corresponding to the highest root,  $\theta$ , of  $A_2$  arises naturally as the level 2 generator  $E^{123456}$ , i.e.

$$e_\theta \equiv E^{123456} = [E^{123}, E^{456}]. \tag{5.17}$$

Although they are guaranteed to hold from the general argument given above, it is instructive and easy to verify explicitly the Serre relations. These read,

$$[e_1, [e_1, e_2]] = [e_2, [e_2, e_1]] = 0 \quad [f_1, [f_1, f_2]] = [f_2, [f_2, f_1]] = 0, \tag{5.18}$$



**Figure 7:**  $(6_2, 4_3)$ : The first configuration with intersecting lines.

and are satisfied since the level 3-generators are killed because of antisymmetry, e.g.

$$[E^{123}, [E^{123}, E^{456}]] = [E^{123}, E^{123456}] = E^{1|23123456} + E^{2|31123456} + E^{3|12123456} = 0, \quad (5.19)$$

where each generator in the last step vanished individually.

Since the Killing form on the Cartan subalgebra of  $A_2$  has Euclidean signature, one must extend  $A_2$  by an appropriate one-dimensional timelike subalgebra  $\mathbb{R}l$  of  $\mathfrak{h}_{E_{10}}$ . We take  $l = K^7_7 + K^8_8 + K^9_9 + K^{10}_{10}$ . In  $A_2 \oplus \mathbb{R}l$ , the Hamiltonian constraint can be fulfilled. Furthermore, since  $A_2 \oplus \mathbb{R}l$  has generators only up to level two, the  $\sigma$ -model equations of motion are equivalent to the dynamical supergravity equations without need to implement an additional level truncation.

The fact that there is a level 2 generator implies the generic presence of a non-zero magnetic field. The momentum constraint and Gauss' law are automatically fulfilled because the only non-vanishing components of the 4-form  $F_{\alpha\beta\gamma\delta}$  are  $F_{0123}$ ,  $F_{0456}$  and  $F_{789(10)}$ .

### 5.5 Geometric configuration $(6_2, 4_3)$

We now treat the configuration  $(6_2, 4_3)$ , shown in figure 7. Although the graph is more complicated, the corresponding algebra is actually a lot simpler.

The generators associated to the simple roots are

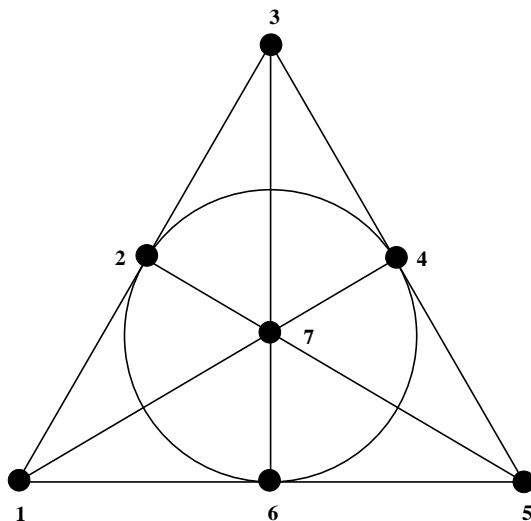
$$e_1 = E^{123} \quad e_2 = E^{145} \quad e_3 = E^{246} \quad e_4 = E^{356}. \quad (5.20)$$

The first thing to note is that in contrast to the previous case, all generators now have one index in common since in the graph any two lines share one node. This implies that the 4 lines in  $(6_2, 4_3)$  define 4 commuting  $A_1$  subalgebras,

$$(6_2, 4_3) \quad \Longleftrightarrow \quad \mathfrak{g}_{(6_2, 4_3)} = A_1 \oplus A_1 \oplus A_1 \oplus A_1. \quad (5.21)$$

Again, although this is not necessary, one can make sure that the Chevalley-Serre relations are indeed fulfilled. For instance, the Cartan element  $h = [E^{b_1 b_2 b_3}, F_{b_1 b_2 b_3}]$  (no summation on the fixed, distinct indices  $b_1, b_2, b_3$ ) reads

$$h = -\frac{1}{3} \sum_{a \neq b_1, b_2, b_3} K^a_a + \frac{2}{3} (K^{b_1}_{b_1} + K^{b_2}_{b_2} + K^{b_3}_{b_3}). \quad (5.22)$$



**Figure 8:**  $(7_3, 7_3)$ : The *Fano plane*, dual to the Lie algebra  $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$ .

Hence, the commutator  $[h, E^{b_i cd}]$  vanishes whenever  $E^{b_i cd}$  has only one  $b$ -index,

$$\begin{aligned}
 [h, E^{b_i cd}] &= -\frac{1}{3}[(K^c_c + K^d_d), E^{b_i cd}] + \frac{2}{3}[(K^{b_1}_{b_1} + K^{b_2}_{b_2} + K^{b_3}_{b_3}), E^{b_i cd}] \\
 &= \left(-\frac{1}{3} - \frac{1}{3} + \frac{2}{3}\right)E^{b_i cd} = 0 \quad (i = 1, 2, 3).
 \end{aligned}
 \tag{5.23}$$

Furthermore, multiple commutators of the step operators are immediately killed at level 2 whenever they have one index or more in common, e.g.

$$[E^{123}, E^{145}] = E^{123145} = 0.
 \tag{5.24}$$

To fulfill the Hamiltonian constraint, one must extend the algebra by taking a direct sum with  $\mathbb{R}l$ ,  $l = K^7_7 + K^8_8 + K^9_9 + K^{10}_{10}$ . Accordingly, the final algebra is  $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}l$ . Because there is no magnetic field, the momentum constraint and Gauss' law are identically satisfied.

The gravitational solution associated to this configuration generalizes the one found in [1]. In fact, using the terminology of [37], the solution describes a set of four intersecting *SM2*-branes, with a five-dimensional transverse spacetime in the directions  $t, x^7, x^8, x^9, x^{10}$ . We postpone a more detailed discussion of this solution to section 8.

### 5.6 Geometric configuration $(7_3, 7_3)$

We now turn to the only existing configuration for  $n = 7$ , which has seven lines and it accordingly denoted  $(7_3, 7_3)$ . The graph is shown in figure 8. Readers familiar with the octonions will recognize this as the so-called *Fano plane*, encoding the complete multiplication table of the octonions (see e.g. [34, 35] for an introduction).

We see from figure 8 that any two lines have exactly one node in common and hence the corresponding algebras will necessarily be commuting. Since the graph has 7 lines we

conclude that the Fano plane is dual to the direct sum of seven  $A_1$  algebras,

$$(7_3, 7_3) \iff \mathfrak{g}_{\text{Fano}} = A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1. \quad (5.25)$$

Because there are seven points, the algebra is in fact embedded in  $E_7$ . Note that the  $A_1$ 's are NOT the  $\mathfrak{sl}(2)$  subalgebras associated with the simple roots of  $E_7$  since the  $A_1$ 's in  $\mathfrak{g}_{\text{Fano}}$  are commuting.

Although of rank 7, the regular embedding of  $\mathfrak{g}_{\text{Fano}}$  in  $E_7$  is not maximal, but is part of the following chain of maximal regular embeddings:

$$A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \subset A_1 \oplus A_1 \oplus A_1 \oplus D_4 \subset A_1 \oplus D_6 \subset E_7 \quad (5.26)$$

as can be verified by using the Dynkin argument based on the highest root [26]. The intermediate algebras occurring in (5.26) have as raising operators (with the choice of  $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$  captured by figure 8)  $E^{123}, E^{174}, E^{165}, K^3_2, K^6_5, K^7_4, E^{245}$  (for  $A_1 \oplus A_1 \oplus A_1 \oplus D_4$ ) and  $E^{174}, K^4_1, K^7_4, K^5_3, K^3_2, K^6_5, E^{123}$  (for  $A_1 \oplus D_6$ ).

The algebra being finite-dimensional, one needs to supplement it by a one-dimensional timelike subalgebra  $\mathbb{R}l$  of  $\mathfrak{h}_{E_{10}}$  in order to fulfill the Hamiltonian constraint. One can take  $l = K^8_8 + K^9_9 + K^{10}_{10}$ , which is orthogonal to it. Finally, because there is no level two element, there is again no magnetic field and hence no momentum or Gauss constraint to be concerned about. The solutions of the  $\sigma$ -model for  $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus \mathbb{R}l$  fulfilling the Hamiltonian constraint all define solutions of eleven-dimensional supergravity. Since the  $A_1$ -algebras commute the corresponding solutions describe a set of seven intersecting  $SM2$ -branes.

### 5.7 Geometric configuration $(8_3, 8_3)$

The last finite-dimensional case is provided by the geometric configuration  $(8_3, 8_3)$ . Since there are eight points and eight lines, the dual algebra is a rank-eight algebra regularly embedded in  $E_8$ . Applying the rules derived above to the geometric configuration  $(8_3, 8_3)$ , depicted in figure 9, one easily finds

$$\mathfrak{g}_{(8_3, 8_3)} = A_2 \oplus A_2 \oplus A_2 \oplus A_2. \quad (5.27)$$

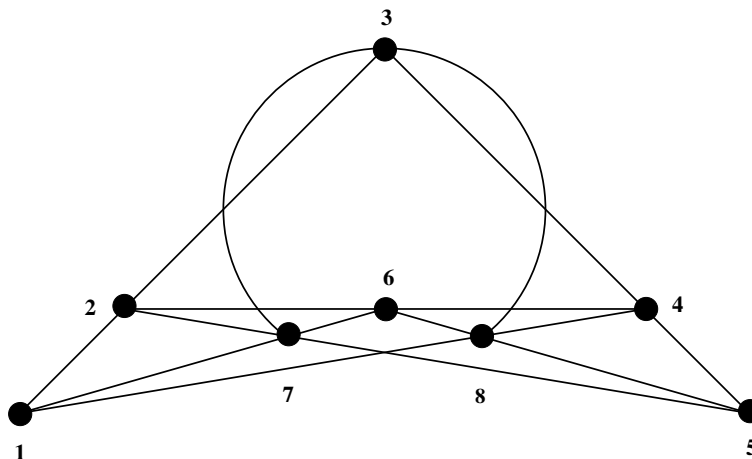
Although this is a rank-eight subalgebra of  $E_8$ , it is not a maximal regular subalgebra, but part of the chain of regular embeddings

$$A_2 \oplus A_2 \oplus A_2 \oplus A_2 \subset A_2 \oplus E_6 \subset E_8. \quad (5.28)$$

With the numbering of the lines of figure 9, the intermediate algebra  $A_2 \oplus E_6$  may be taken to have as raising operators  $E^{123}, E^{568}$  (for  $A_2$ ) and  $K^2_1, K^3_2, K^6_5, K^8_6, K^7_4, E^{145}$  for  $E_6$ .

In order to fulfill the Hamiltonian constraint, we add  $\mathbb{R}l$  with  $l = K^9_9 + K^{10}_{10}$ . The final algebra is thus  $A_2 \oplus A_2 \oplus A_2 \oplus A_2 \oplus \mathbb{R}l$ .

There is no level-3 field in  $A_2 \oplus A_2 \oplus A_2 \oplus A_2$ , so level-3 truncation is automatic in this model. However, because of the level-2 magnetic field generically present, the momentum and Gauss constraints need to be analyzed. The only non-vanishing components of the



**Figure 9:** The geometric configuration  $(8_3, 8_3)$ , dual to the Lie algebra  $A_2 \oplus A_2 \oplus A_2 \oplus A_2$ .

magnetic field arising in the model are  $F_{479(10)}$ ,  $F_{289(10)}$ ,  $F_{369(10)}$  and  $F_{159(10)}$ . Because these have always at least two indices (9 and 10) distinct from the indices on the electric field components, the momentum constraint is satisfied. Furthermore, because they share the pair (9,10), Gauss' law is also fulfilled. Accordingly, the solutions of the  $\sigma$ -model for  $A_2 \oplus A_2 \oplus A_2 \oplus A_2 \oplus \mathbb{R}l$  fulfilling the Hamiltonian constraint all define solutions of eleven-dimensional supergravity.

## 6. Geometric configurations $(9_m, g_3)$

All algebras that arise from  $n = 9$  configurations are naturally embedded in  $E_9$ . They turn out to be infinite dimensional contrary to the cases with  $n \leq 8$ . Furthermore, because they involve, as we shall see, affine algebras and degenerate Cartan matrices, they turn out to be obtained from Kac-Moody algebras through non trivial quotients. In total, there are 7 different configurations with nine nodes, which we consider in turn.

Because the algebras are infinite-dimensional, one must truncate to level 2 in order to match the Bianchi I supergravity equations with the  $\sigma$ -model equations. Furthermore, if taken to be non zero, the magnetic field must fulfill the relevant momentum and Gauss constraints.

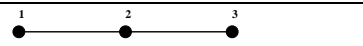
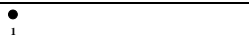
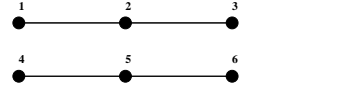
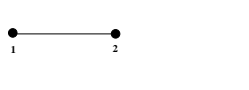
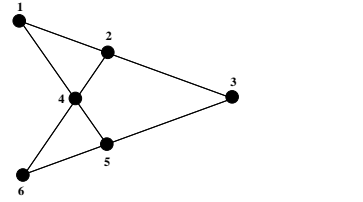
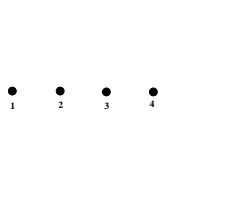
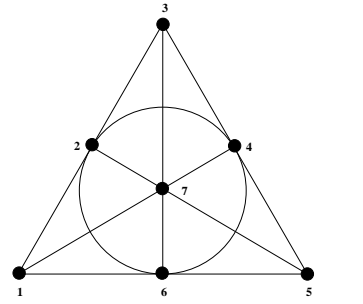
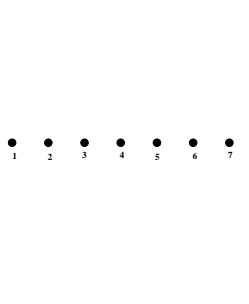
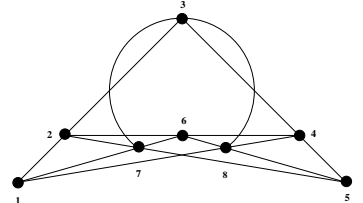
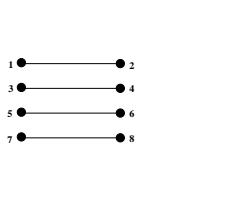
### 6.1 Geometric configurations $(9_1, 3_3)$ and $(9_2, 6_3)$

#### 6.1.1 Geometric configuration $(9_1, 3_3)$

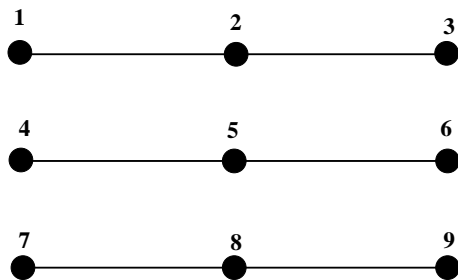
The geometric configuration  $(9_1, 3_3)$  is somewhat trivial and is given in figure 10.

By direct application of the rules, we deduce that the associated Dynkin diagram consists of three nodes (corresponding to the three lines  $\{1, 2, 3\}$ ,  $\{4, 5, 6\}$  and  $\{7, 8, 9\}$ ) that must



Configuration	Dynkin diagram	Lie algebra
		$\mathfrak{g}_{(3_1, 1_3)} = A_1$
		$\mathfrak{g}_{(6_1, 2_3)} = A_2$
		$\mathfrak{g}_{(6_2, 4_3)} = A_1 \oplus A_1 \oplus A_1 \oplus A_1$
		$\mathfrak{g}_{(7_3, 7_3)} = A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1 \oplus A_1$
		$\mathfrak{g}_{(8_3, 8_3)} = A_2 \oplus A_2 \oplus A_2 \oplus A_2$

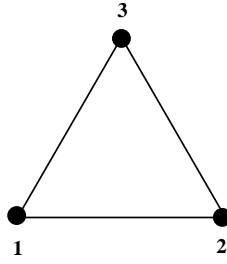
**Table 1:** All configurations for  $n \leq 8$  and their dual finite dimensional Lie algebras.



**Figure 10:** The geometric configuration  $(9_1, 3_3)$ .

all be connected since the corresponding lines in the configuration are parallel. This gives the Cartan matrix of  $A_2^+$ , i.e. the untwisted affine extension of  $A_2$ ,

$$A(A_2^+) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \tag{6.1}$$



**Figure 11:** The Dynkin diagram of  $A_2^+$ .

whose Dynkin diagram is shown in figure 11. The Cartan matrix of  $A_2^+$  is degenerate and so, it is not guaranteed that the algebra generated by the raising operators

$$e_1 = E^{123} \quad e_2 = E^{456} \quad e_3 = E^{789}, \quad (6.2)$$

the lowering operators

$$f_1 = F_{123} \quad f_2 = F_{456} \quad f_3 = F^{789}, \quad (6.3)$$

and the corresponding Cartan elements given by eq. (5.22),

$$h_1 = [e_1, f_1] = -\frac{1}{3} \sum_{a \neq 1,2,3} K^a_a + \frac{2}{3} (K^1_1 + K^2_2 + K^3_3), \quad (6.4)$$

$$h_2 = [e_2, f_2] = -\frac{1}{3} \sum_{a \neq 4,5,6} K^a_a + \frac{2}{3} (K^4_4 + K^5_5 + K^6_6), \quad (6.5)$$

$$h_3 = [e_3, f_3] = -\frac{1}{3} \sum_{a \neq 7,8,9} K^a_a + \frac{2}{3} (K^7_7 + K^8_8 + K^9_9), \quad (6.6)$$

is the untwisted Kac-Moody algebra  $A_2^+$ .

To investigate this issue, we recall some properties of untwisted affine Kac-Moody algebras. Consider a finite-dimensional, simple Lie algebra  $\mathfrak{g}$ . One can associate with it three related infinite-dimensional Lie algebras:

- the (untwisted) affine Kac-Moody algebra  $\mathfrak{g}^+$ ,
- the current algebra  $\mathfrak{g}^J$ ,
- the loop algebra  $\tilde{\mathfrak{g}}$ .

If  $\{T_A\}$  is a basis of  $\mathfrak{g}$  with structure constants  $C^A_{BC}$ , the loop algebra  $\tilde{\mathfrak{g}}$  has basis  $\{T_A^n\}$  ( $n \in \mathbb{Z}$ ) with commutation relations

$$[T_B^n, T_C^m] = C^A_{BC} T_A^{m+n}, \quad (6.7)$$

the current algebra  $\mathfrak{g}^J$  has basis  $\{T_A, c\}$  with commutation relations

$$[T_B^n, T_C^m] = C^A_{BC} T_A^{m+n} + n \delta_{n+m,0} k_{BC} c, \quad [T_A^n, c] = 0, \quad (6.8)$$

while the Kac-Moody algebra  $\mathfrak{g}^+$  has basis  $\{T_A, c, d\}$  with commutation relations

$$[T_B^n, T_C^m] = C^A{}_{BC} T_A^{m+n} + n \delta_{n+m,0} k_{BC} c, \quad [T_A^n, c] = 0, \quad (6.9)$$

$$[d, T_A^n] = n T_A^n, \quad [d, c] = 0. \quad (6.10)$$

Here,  $k_{AB}$  is an invariant form (Killing form) on  $\mathfrak{g}$ . As vector spaces,  $\mathfrak{g}^+ = \mathfrak{g}^J \oplus \mathbb{R}d = \tilde{\mathfrak{g}} \oplus \mathbb{R}c \oplus \mathbb{R}d$  and  $\mathfrak{g}^J = \tilde{\mathfrak{g}} \oplus \mathbb{R}c$ . The algebra  $\mathfrak{g}^+$  is the Kac-Moody algebra associated to the Cartan matrix  $A_{ij}^+$  obtained from the Cartan matrix of  $\mathfrak{g}$  by adding minus the affine root. Because the Cartan matrix  $A_{ij}^+$  has vanishing determinant, the construction of  $\mathfrak{g}^+$  involves a non trivial “realization of  $A_{ij}^+$ ” [18], which is how the scaling operator  $d$  enters.

The operator  $d$  is in the Cartan subalgebra of  $\mathfrak{g}^+$  and has the following scalar products with all the Cartan generators [18],

$$\begin{cases} \langle d|h_a \rangle = 0 & (\text{for } h_a \text{ in the Cartan subalgebra of } \mathfrak{g}) \\ \langle d|d \rangle = 0 \\ \langle d|c \rangle = 1. \end{cases} \quad (6.11)$$

Note also that  $\langle c|c \rangle = 0$ . The root lattice,  $\Lambda_{\mathfrak{g}^+}$ , of  $\mathfrak{g}^+$  is constructed by adding to the root lattice of  $\mathfrak{g}$  a null vector  $\delta \in \Pi_{1,1}$ , where  $\Pi_{1,1}$  is the 2-dimensional self-dual Lorentzian lattice [18]. Thus the root lattice of  $\mathfrak{g}^+$  is contained in the direct sum of  $\Lambda_{\mathfrak{g}}$  with  $\Pi_{1,1}$ , i.e.

$$\Lambda_{\mathfrak{g}^+} \subset \Lambda_{\mathfrak{g}} \oplus \Pi_{1,1}. \quad (6.12)$$

The affine root is given by

$$\alpha_0 \equiv \delta - \theta, \quad (6.13)$$

where  $\theta$  is the highest root of  $\mathfrak{g}$ . The scaling generator  $d$  counts the number of times the raising operator corresponding to the affine root  $\alpha_0$  appears in any multiple commutator in  $\mathfrak{g}^+$ .

When  $\mathfrak{g}$  is simple, as is the algebra  $A_2$  relevant for the  $(9_1, 3_3)$  configuration, the current algebra  $\mathfrak{g}^J$  is the derived algebra of  $\mathfrak{g}^+$ ,  $\mathfrak{g}^J = [\mathfrak{g}^+, \mathfrak{g}^+] \equiv (\mathfrak{g}^+)'$ . The current algebra coincides with the algebra generated by the Chevalley-Serre relations associated with the given Cartan matrix  $A_{ij}^+$ , and not with its realization. Furthermore, the center of  $\mathfrak{g}^J$  and  $\mathfrak{g}^+$  is one-dimensional and given by  $\mathbb{R}c$ . The loop algebra  $\tilde{\mathfrak{g}}$  is the quotient of the current algebra  $\mathfrak{g}^J$  by its one-dimensional center  $\mathbb{R}c$ .

In fact, according to Theorem 1.7 in [18], the only ideals of a Kac-Moody algebra  $\mathfrak{a}$  with non-decomposable Cartan matrix either contain its derived algebra  $\mathfrak{a}'$  or are contained in its center.

In the case of the configuration  $(9_1, 3_3)$ , we have all the generators  $\{h_i, e_i, f_i\}$  fulfilling the Chevalley-Serre relations associated with the Cartan matrix of  $A_2^+$ , without enlargement of the Cartan subalgebra to contain the scaling operator  $d$ . Hence, the algebra dual to  $(9_1, 3_3)$  must either be the current algebra  $A_2^J$  or its quotient by its center - the loop algebra  $\tilde{A}_2$ . This would be the case if the center were represented trivially. But the central charge is not trivial and given by

$$c \equiv h_1 + h_2 + h_3 = -K^1{}_{10}, \quad (6.14)$$

which does not vanish. Hence, the relevant algebra is the current algebra  $A_2^J \equiv (A_2^+)'$ .

It is straightforward to verify that  $c$  commutes with all the generators of  $A_2^J$ . It is also possible to define, within the Cartan subalgebra of  $E_{10}$ , an element  $d$  that plays the role of a scaling operator. This enlargement of  $(A_2^+)'$  leads to the full Kac-Moody algebra  $A_2^+$ . It is necessary in order to have a scalar product on the Cartan subalgebra which is of Lorentzian signature, as required if one wants to solve the Hamiltonian constraint within the algebra.

Choosing  $E^{123}$  as the “affine” generator, there exists a six-parameter family of scaling operators,

$$d = a_1 K^1_1 + a_2 K^2_2 + a_3 K^3_3 + b_1 K^4_4 + b_2 K^5_5 + b_3 K^6_6 + c_1 K^7_7 + c_2 K^8_8 + c_3 K^9_9 + p K^{10}_{10} \tag{6.15}$$

with

$$\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ b_1 + b_2 + b_3 &= c_1 + c_2 + c_3 = 0 \\ a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 + c_1^2 + c_2^2 + c_3^2 &= 1 - 2p. \end{aligned} \tag{6.16}$$

The simplest and most convenient choice is to take for  $d$ ,

$$d = K^1_1. \tag{6.17}$$

Let us check that the null generator  $e_\delta$ ,

$$e_\delta \equiv [E^{123}, [E^{456}, E^{789}]], \tag{6.18}$$

associated with the null root

$$\delta = \alpha_0 + \theta = \alpha_1 + \alpha_2 + \alpha_3, \tag{6.19}$$

where  $\theta = \alpha_2 + \alpha_3$  is the highest root of  $A_2$ , is indeed an eigenvector with eigenvalue 1 for the adjoint action of  $d$ . To this end we observe that

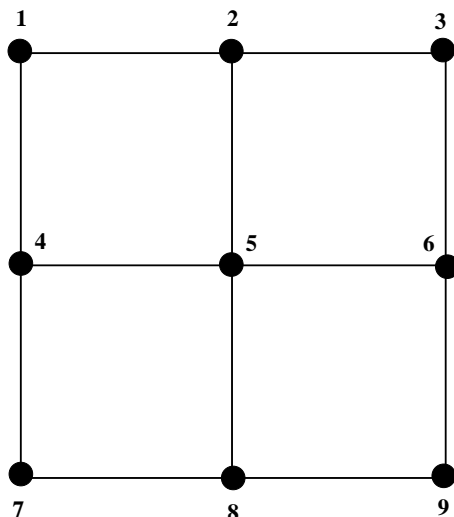
$$[d, E^{123}] = [K^1_1, E^{123}] = E^{123}, \quad [d, E^{456}] = [d, E^{789}] = 0. \tag{6.20}$$

Hence,  $d$  counts indeed the number of times  $E^{123}$  appears in any commutator so that one gets

$$[d, e_\delta] = e_\delta, \tag{6.21}$$

as desired.

Note that among the momentum constraints and Gauss’ law, the only non identically vanishing condition on the magnetic field is the 10-th component of the momentum constraint.



**Figure 12:** The geometric configuration  $(9_2, 6_3)_1$ .

**6.1.2 Geometric configurations  $(9_2, 6_3)$**

There are two geometric configurations  $(9_2, 6_3)$ . We start with  $(9_2, 6_3)_1$ , shown in figure 12. The configuration consists of two sets with three distinct triples in each set:  $S_{(1)} = \{(123), (456), (789)\}$ ,  $S_{(2)} = \{(147), (258), (369)\}$ . By direct application of the rules from above we can state that all generators associated with each set will commute with the generators from the other set. Thus the corresponding Cartan matrix is decomposable and equal to the direct sum of two  $A_2^+$ 's. This  $6 \times 6$  matrix has rank 4.

Since the generators  $\{h_i, e_i, f_i\}$  ( $i = 1, \dots, 6$ ) of  $\mathfrak{g}_{(9_2, 6_3)_1}$  fulfill the Chevalley-Serre relations associated with the matrix  $A_2^+ \oplus A_2^+$  (un-enlarged, i.e., not its realization), the algebra  $\mathfrak{g}_{(9_2, 6_3)_1}$  dual to  $(9_2, 6_3)_1$  is either the derived algebra  $(A_2^+ \oplus A_2^+)' = (A_2^+)' \oplus (A_2^+)'$  or a quotient of this algebra by a subspace of its center. The center of  $(A_2^+)' \oplus (A_2^+)'$  is two-dimensional and generated by the two central charges  $c_{(1)} = h_1 + h_2 + h_3$  and  $c_{(2)} = h_4 + h_5 + h_6$ . It is clear that these central charges are not independent in the algebra  $\mathfrak{g}_{(9_2, 6_3)_1}$  since

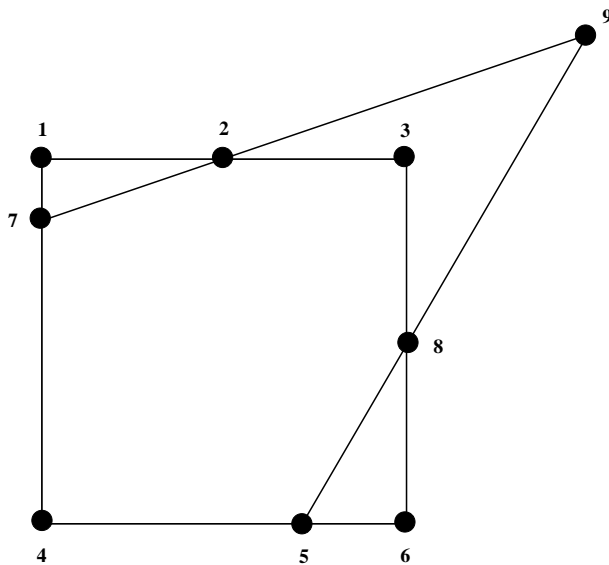
$$c_{(1)} = c_{(2)} = -K^{10}_{10}. \tag{6.22}$$

The two  $(A_2^+)'$ 's share therefore the same central charge. Hence  $\mathfrak{g}_{(9_2, 6_3)_1}$  is the quotient of  $(A_2^+)' \oplus (A_2^+)'$  by the ideal  $\mathbb{R}(c_{(1)} - c_{(2)})$ ,

$$\mathfrak{g}_{(9_2, 6_3)_1} = \frac{(A_2^+)' \oplus (A_2^+)' }{\mathbb{R}(c_{(1)} - c_{(2)})}. \tag{6.23}$$

This is the current algebra  $(A_2 \oplus A_2)^J$  of  $A_2 \oplus A_2$  with a single central charge ((6.8) with a single  $c$ ).

We can again introduce a (single) scaling element within the Cartan subalgebra of  $E_{10}$ . Taking the affine roots to be  $\alpha_{123}$  and  $\alpha_{147}$  (with generators  $E^{123}$  and  $E^{147}$ ), one may



**Figure 13:** The geometric configuration  $(9_2, 6_3)_2$ .

choose

$$d = K^1_1, \tag{6.24}$$

as before. In the algebra enlarged with the scaling operator, the Hamiltonian constraint can be satisfied since the metric in the Cartan subalgebra has Lorentzian signature.

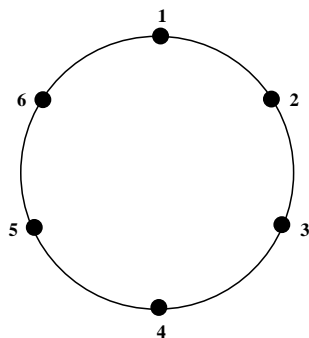
An interesting new phenomenon occurs also for this configuration, namely that the null roots of both algebras are equal (and equal to  $\beta^1 + \beta^2 + \beta^3 + \beta^4 + \beta^5 + \beta^6 + \beta^7 + \beta^8 + \beta^9$  in the billiard parametrization). Hence the vector space spanned by the roots of  $\frac{(A_2^+)' \oplus (A_2^+)'}{\mathbb{R}(c_{(1)} - c_{(2)})}$  in the space of the roots of  $E_9$  is 5-dimensional. This “disappearance of one dimension” is compatible with the fact that both null roots have the same scaling behaviour under  $d$  and is possible because we do not have an embedding of the full Kac-Moody algebra  $A_2^+ \oplus A_2^+$  with two independent scaling operators under which the two null roots behave distinctly. Note that, of course, the corresponding generators  $[E^{123}, [E^{456}, E^{789}]]$  and  $[E^{147}, [E^{258}, E^{369}]]$  are linearly independent.

The other configuration  $(9_2, 6_3)$  is the configuration  $(9_2, 6_3)_2$ , depicted in figure 13. The analysis proceeds as for the configuration  $(9_1, 3_3)$ . The computation of the Cartan matrix is direct and yields the Dynkin diagram shown in figure 14, which is recognized as being the diagram of the untwisted affine extension  $A_5^+$  of  $A_5$ .

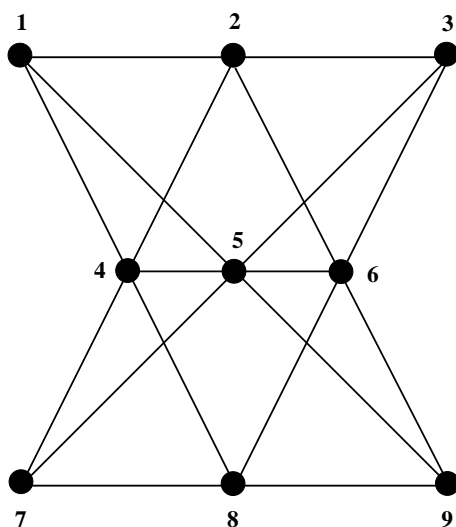
The dual algebra is now the current algebra  $(A_5^+)',$  with central charge  $c = -K^{10}_{10}$ . If one regards  $\alpha_{123}$  as the affine root, one can add the scaling operator  $d = K^1_1$  to get the complete Kac-Moody algebra  $A_5^+.$

### 6.2 Geometric configurations $(9_3, 9_3)$

There are three geometric configurations  $(9_3, 9_3)$ . Their treatment is a direct generalization



**Figure 14:** The Dynkin diagram of  $A_5^+$ , dual to the geometric configuration  $(9_2, 6_3)_2$ .



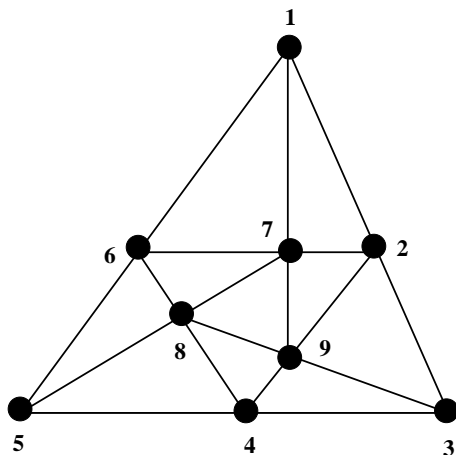
**Figure 15:**  $(9_3, 9_3)_1$ : This is the so-called *Pappus configuration*.

of what we have discussed before. Let us consider first the configuration  $(9_3, 9_3)_1$ , displayed in figure 15. This configuration was constructed by *Pappus of Alexandria* during the 3<sup>rd</sup> century A.D. with the purpose of illustrating the following theorem (adapted from [17]):

*Let three points  $\{1, 2, 3\}$  lie in consecutive order on a single straight line and let three other points  $\{7, 8, 9\}$  lie in consecutive order on another straight line. Then the three pairwise intersections  $4 = \{1, 2\} \cap \{7, 8\}$ ,  $5 = \{1, 3\} \cap \{7, 9\}$  and  $6 = \{2, 3\} \cap \{8, 9\}$  are collinear.*

The configuration is also called the Brianchon-Pascal configuration [16].

By inspecting the Pappus configuration we note that it consists of three sets with three distinct triples in each set:  $S_1 = \{(123), (456), (789)\}$ ,  $S_2 = \{(159), (368), (247)\}$ ,  $S_3 = \{(269), (357), (148)\}$ . Hence its Cartan matrix is decomposable and the direct sum of three times the Cartan matrix  $A_2^+$  of the untwisted affine extension of  $A_2$ . It has rank



**Figure 16:** The geometric configuration  $(9_3, 9_3)_2$ .

6. As in the previous examples, this does not imply, however, that the complete algebra associated to the Pappus configuration is a direct sum of  $A_2^+$  algebras, or the derived algebra. One has non trivial quotients because the three  $(A_2^+)'$  share the same central charge. Indeed, one finds again, just as above, the relation

$$c_{(1)} = c_{(2)} = c_{(3)} = -K^{10}_{10}. \tag{6.25}$$

Hence  $\mathfrak{g}_{\text{Pappus}} \equiv \mathfrak{g}_{(9_3, 9_3)_1}$  is the quotient of  $(A_2^+)' \oplus (A_2^+)' \oplus (A_2^+)'$  by the ideal  $\mathbb{R}(c_{(1)} - c_{(2)}) \oplus \mathbb{R}(c_{(1)} - c_{(3)})$ , i.e., the current algebra  $(A_2 \oplus A_2 \oplus A_2)^J$  with only one central charge,

$$\mathfrak{g}_{\text{Pappus}} = \frac{(A_2^+)' \oplus (A_2^+)' \oplus (A_2^+)' }{\mathbb{R}(c_{(1)} - c_{(2)}) \oplus \mathbb{R}(c_{(1)} - c_{(3)})} = (A_2 \oplus A_2 \oplus A_2)^J. \tag{6.26}$$

Regarding the affine roots as being  $\alpha_{123}$ ,  $\alpha_{159}$  and  $\alpha_{148}$ , one can add to the algebra the scaling operator  $d = K^1_1$ , a task necessary to be able to get non trivial solutions of the Hamiltonian constraint within the algebra.

The two remaining  $n = 9$  configurations with 9 lines are shown in figures 16 and 17, respectively.

The configuration  $(9_3, 9_3)_2$  leads to the Dynkin diagram of  $A_8^+$  and to the derived algebra  $(A_8^+)' \equiv A_8^J$ . Taking  $\alpha_{123}$  as the affine root, one can add the scaling element

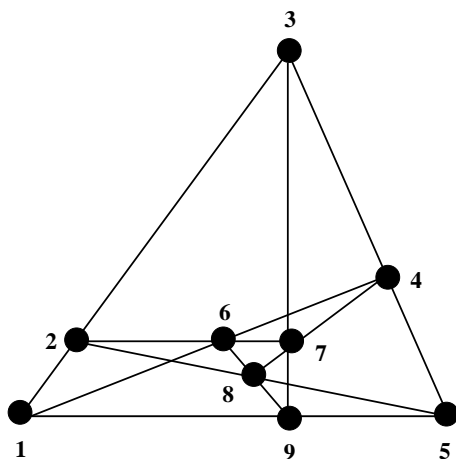
$$d = \frac{1}{6}(2K^1_1 + 2K^2_2 + 2K^3_3 - K^4_4 - K^5_5 - K^6_6 - K^7_7 + 2K^8_8 - K^9_9 - 2K^{10}_{10}) \tag{6.27}$$

to get the complete Kac-Moody algebra  $A_8^+$ .

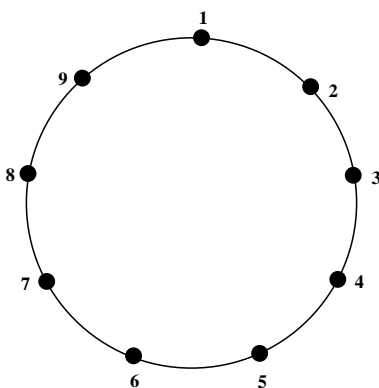
The configuration  $(9_3, 9_3)_3$  leads to the Dynkin diagram of  $A_2^+ \oplus A_5^+$  and to the algebra  $\frac{(A_2^+)' \oplus (A_5^+)' }{\mathbb{R}(2c_{(1)} - c_{(2)})}$  with only one central charge. Taking  $\alpha_{123}$  and  $\alpha_{146}$  as the affine roots, there exists a one-parameter family of scaling operators of the form

$$d = \frac{1}{6}(5K^1_1 + 4K^2_2 + 3K^3_3 + 2K^4_4 - 5K^5_5 - K^6_6 - 3K^7_7 + K^8_8)$$





**Figure 17:** The geometric configuration  $(9_3, 9_3)_3$ .



**Figure 18:** The Dynkin diagram of  $A_8^+$  associated with the configuration  $(9_3, 9_3)_2$ .

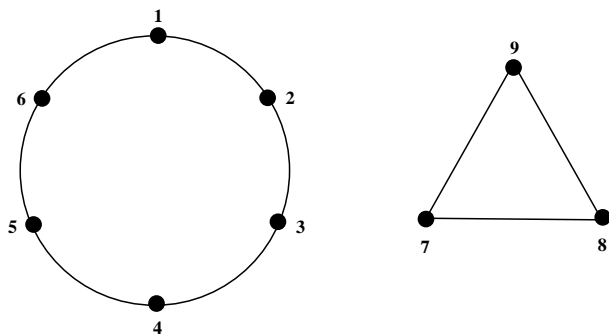
$$\begin{aligned}
 & + \frac{p}{2}(-K^1_1 + 2K^2_2 - K^3_3 + 2K^4_4 - K^5_5 - K^6_6 - K^7_7 - K^8_8 + 2K^9_9) \\
 & + \frac{1}{4}(3 + 4p + 9p^2)K^{10}_{10}.
 \end{aligned} \tag{6.28}$$

Note that this operator actually counts the number of roots  $\alpha_{146}$  but twice the number of roots  $\alpha_{123}$ . This is in accordance with the fact that the ideal is of the form  $\mathbb{R}(2c_{(1)} - c_{(2)})$ . The corresponding Dynkin diagrams are shown in figures 18 and 19, respectively.

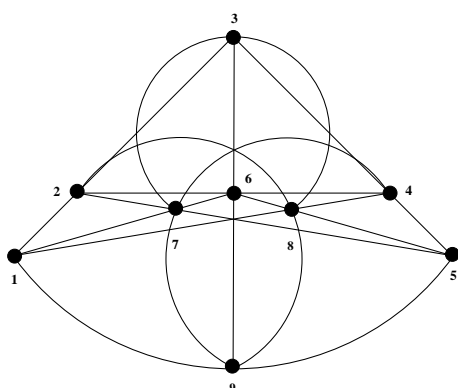
### 6.3 Geometric configuration $(9_4, 12_3)$

The geometric configuration is shown in figure 20. We find the Dynkin diagram of  $A_2^+ \oplus A_2^+ \oplus A_2^+ \oplus A_2^+$ . The relevant algebra is then the direct sum of the corresponding derived algebras with same central charge, i.e.

$$\mathfrak{g}_{(9_4, 12_3)} = \frac{(A_2^+)' \oplus (A_2^+)' \oplus (A_2^+)' \oplus (A_2^+)'}{\mathbb{R}(c_{(1)} - c_{(2)}) \oplus \mathbb{R}(c_{(1)} - c_{(3)}) \oplus \mathbb{R}(c_{(1)} - c_{(4)})}. \tag{6.29}$$



**Figure 19:** The Dynkin diagram of  $A_2^+ \oplus A_5^+$  associated with the configuration  $(9_3, 9_3)_3$ .

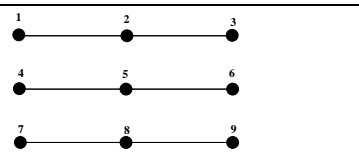
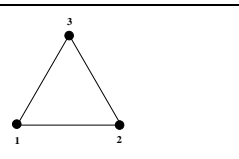
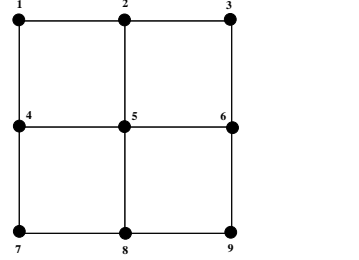
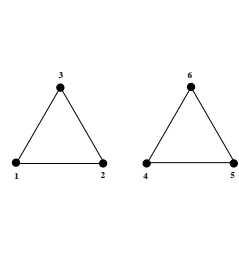
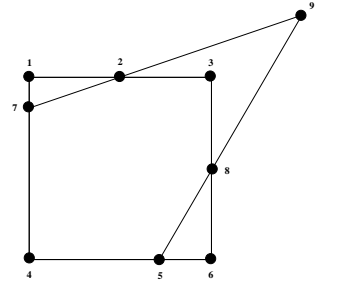
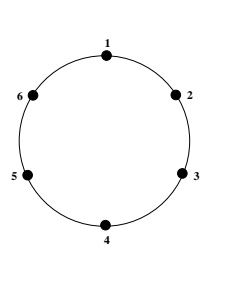
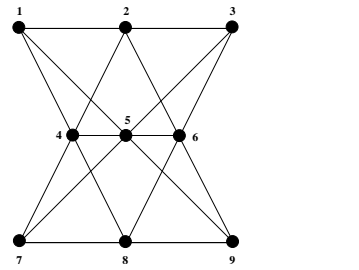
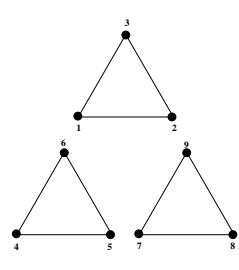
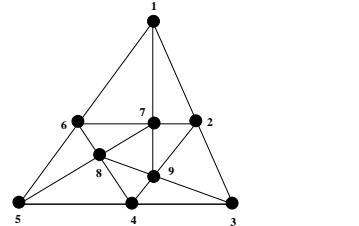
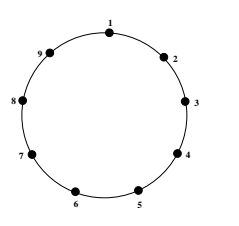


**Figure 20:** The geometric configuration  $(9_4, 12_3)$ .

The scaling operator  $d = K^{-1}_1$  can be added to the algebra.

This result on  $(9_4, 12_3)$  is intimately connected with the analysis of the geometric configuration  $(8_3, 8_3)$ , for which the algebra is  $A_2 \oplus A_2 \oplus A_2 \oplus A_2$ . This algebra can be embedded in  $E_8$  and, accordingly, the corresponding current algebra with a single central charge can be embedded in the current algebra  $E'_9 \equiv (E_8^+)'$  of  $E_8$ . On the side of the geometric configurations, the affinization of  $A_2 \oplus A_2 \oplus A_2 \oplus A_2$  corresponds to adding one point, say 9, to  $(8_3, 8_3)$  and drawing the four lines connecting this new point to the four pairs of unconnected points of  $(8_3, 8_3)$ . This yields  $(9_4, 12_3)$ . Note that this is the only case for which it is possible to extend an  $n = p$  configuration to an  $n = p + 1$  configuration through the inclusion of an additional point directly in the configuration.

The configuration  $(9_4, 12_3)$  also has an interesting interpretation in terms of points of inflection of third-order plane curves [16].

	Configuration	Dynkin diagram	Lie algebra
$(9_1, 3_3)$			$\mathfrak{g}_{(9_1, 3_3)} = A_2^J$
$(9_2, 6_3)_1$			$\mathfrak{g}_{(9_2, 6_3)_1} = (A_2 \oplus A_2)^J$
$(9_2, 6_3)_2$			$\mathfrak{g}_{(9_2, 6_3)_2} = A_5^J$
$(9_3, 9_3)_1$			$\mathfrak{g}_{(9_3, 9_3)_1} = (A_2 \oplus A_2 \oplus A_2)^J$
$(9_3, 9_3)_2$			$\mathfrak{g}_{(9_3, 9_3)_2} = A_8^J$

**Table 2:**  $n = 9$  configurations and their dual affine Kac-Moody algebras.

### 7. Geometric configurations $(10_m, g_3)$

As we shall see, subalgebras constructed from configurations with ten nodes give rise to Lorentzian subalgebras of  $E_{10}$ , except in two cases, denoted  $(10_3, 10_3)_4$  and  $(10_3, 10_3)_7$  in tables 4 and 5 and in [1], for which the Cartan matrix has zero determinant.

Because the rank-10 algebras are infinite-dimensional, one must again truncate to level 2 in order to match the Bianchi I supergravity equations with the  $\sigma$ -model equations. Furthermore, if taken to be non zero, the magnetic field must fulfill the relevant momentum and Gauss constraints.

	Configuration	Dynkin diagram	Lie algebra
$(9_3, 9_3)_3$			$\mathfrak{g}_{(9_3, 9_3)_3} = (A_5 \oplus A_2)^J$
$(9_4, 12_3)$			$\mathfrak{g}_{(9_4, 12_3)} = (A_2 \oplus A_2 \oplus A_2 \oplus A_2)^J$

**Table 3:**  $n = 9$  configurations and their dual affine Kac-Moody algebras.

### 7.1 The Petersen algebra

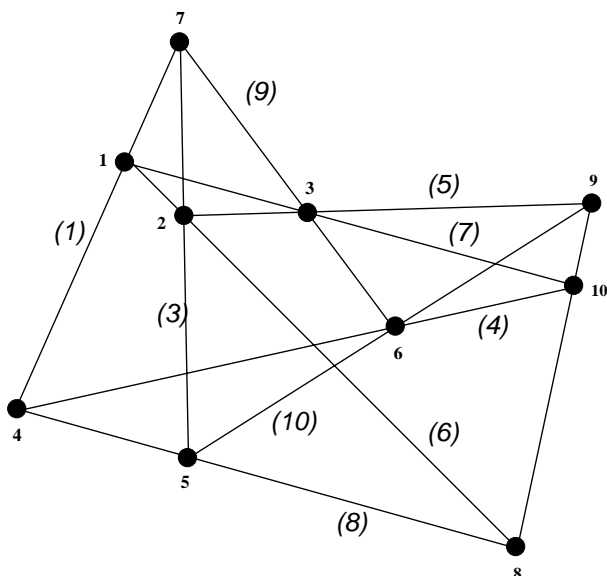
We first illustrate the simple situation with a non degenerate Cartan matrix, for which the above theorem of section 4 applies directly. We consider explicitly the well-known *Desargues configuration*, denoted  $(10_3, 10_3)_3$ , for which a new fascinating feature emerges, namely that the Dynkin diagram dual to it *also* corresponds in itself to a geometric configuration. In fact, the dual Dynkin diagram turns out to be the famous *Petersen graph*, denoted  $(10_3, 15_2)$ . These are displayed in figures 21 and 22, respectively.

The configuration  $(10_3, 10_3)_3$  is associated with the 17<sup>th</sup> century French mathematician *Gérard Desargues* to illustrate the following “Desargues theorem” (adapted from [17]):

*Let the three lines defined by  $\{4, 1\}, \{5, 2\}$  and  $\{6, 3\}$  be concurrent, i.e. be intersecting at one point, say  $\{7\}$ . Then the three intersection points  $8 \equiv \{1, 2\} \cap \{4, 5\}, 9 \equiv \{2, 3\} \cap \{5, 6\}$  and  $10 \equiv \{1, 3\} \cap \{4, 6\}$  are collinear.*

Another way to say this is that the two triangles  $\{1, 2, 3\}$  and  $\{4, 5, 6\}$  in figure 21 are in perspective from the point  $\{7\}$  and in perspective from the line  $\{8, 10, 9\}$ .

To construct the Dynkin diagram we first observe that each line in the configuration is disconnected from three other lines, e.g.  $\{4, 1, 7\}$  have no nodes in common with the lines  $\{2, 3, 9\}, \{5, 6, 9\}, \{8, 10, 9\}$ . This implies that all nodes in the Dynkin diagram will be connected to three other nodes. Proceeding as in the previous section leads to the Dynkin diagram in figure 22, which we identify as the Petersen graph. The corresponding Cartan



**Figure 21:**  $(10_3, 10_3)_3$ : The Desargues configuration, dual to the Petersen graph.

matrix is

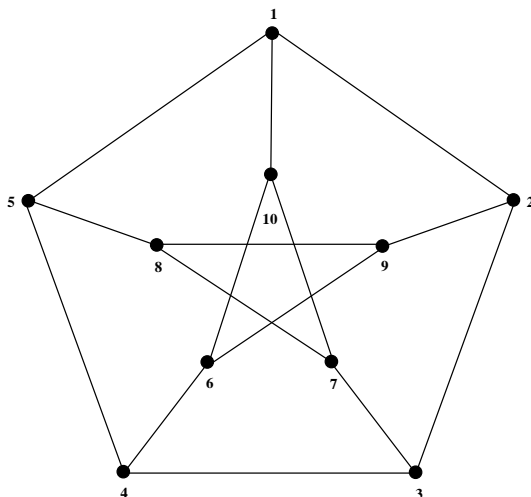
$$A(\mathfrak{g}_{\text{Petersen}}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ -1 & 2 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 2 & 0 \\ -1 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}, \quad (7.1)$$

which is of Lorentzian signature with

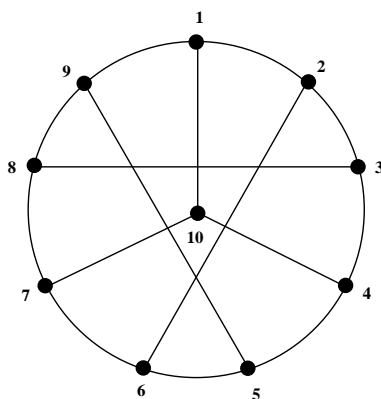
$$\det A(\mathfrak{g}_{\text{Petersen}}) = -256. \quad (7.2)$$

The Petersen graph was invented by the Danish mathematician *Julius Petersen* in the end of the 19<sup>th</sup> century. It has several embeddings on the plane, but perhaps the most famous one is as a star inside a pentagon as depicted in figure 22. One of its distinguishing features from the point of view of graph theory is that it contains a *Hamiltonian path* but no *Hamiltonian cycle*.<sup>7</sup> Because the algebra is Lorentzian (with a metric that coincides with the metric induced from the embedding in  $E_{10}$ ), it does not need to be enlarged by any further generator to be compatible with the Hamiltonian constraint.

<sup>7</sup>We recall that a Hamiltonian path is defined as a path in an undirected graph which intersects each node once and only once. A Hamiltonian cycle is then a Hamiltonian path which also returns to its initial node.



**Figure 22:** This is the so-called *Petersen graph*. It is the Dynkin diagram dual to the Desargues configuration, and is in fact a geometric configuration itself, denoted  $(10_3, 15_2)$ .



**Figure 23:** An alternative drawing of the Petersen graph in the plane. This embedding reveals an  $S_3$  permutation symmetry about the central point. Yet another drawing is given in tables 4 and 5 summarizing the results for the  $(10_3, 10_3)$  configurations.

It is interesting to examine the symmetries of the various embeddings of the Petersen graph in the plane and the connection to the Desargues configurations. The embedding in figure 22 clearly exhibits a  $\mathbb{Z}_5 \times \mathbb{Z}_2$ -symmetry, while the Desargues configuration in figure 21 has only a  $\mathbb{Z}_2$ -symmetry. Moreover, the embedding of the Petersen graph shown in figure 23 reveals yet another symmetry, namely an  $S_3$  permutation symmetry about the central point, labeled “10”. In fact, the external automorphism group of the Petersen graph is  $S_5$  so what we see in the various embeddings are simply subgroups of  $S_5$  made manifest. It is not clear how these symmetries are realized in the Desargues configuration that seems to exhibit much less symmetry.

## 7.2 A degenerate case

We now discuss another interesting case, which requires a special treatment because the corresponding Cartan matrix is degenerate. It is the configuration  $(10_3, 10_3)_4$ , shown in figure 24. By application of the rules, with the generators chosen according to the numbering of the lines in figure 24, i.e.  $(1) = 123, (2) = 456 \dots$  etc, we find that this configuration gives rise to the Dynkin diagram shown in figure 25. The Cartan matrix takes the form

$$A(\mathfrak{g}_{(10_3, 10_3)_4}) = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}, \quad (7.3)$$

which has vanishing determinant

$$\det A(\mathfrak{g}_{(10_3, 10_3)_4}) = 0. \quad (7.4)$$

This Cartan matrix has one negative and one null eigenvalue while the rest of the eigenvalues are positive. Hence, the algebra is indefinite type. The eigenvector associated to the null eigenvalue is given explicitly by

$$u = (0, 1, 1, 1, 0, -1, -1, -1, 0, 0). \quad (7.5)$$

We then deduce that the corresponding algebra has a one dimensional non-trivial center,  $\mathfrak{r} = \{k\}$ , with

$$k = \sum_{i=1}^{10} u_i h_i = -h_2 - h_3 - h_4 + h_6 + h_7 + h_8, \quad (7.6)$$

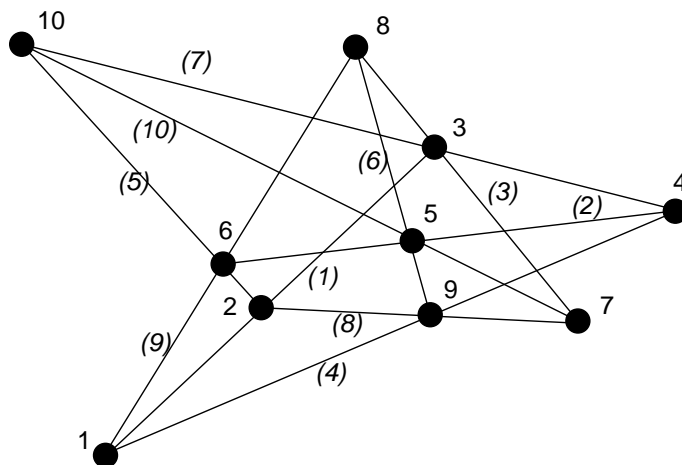
where  $u_i$  are the components of the null eigenvector and  $h_i$  are the generators of the Cartan subalgebra. Making use of the explicit form of  $h_i$ , eq. (5.22), we find that  $k$  vanishes identically in  $E_{10}$

$$k = -h_2 - h_3 - h_4 + h_6 + h_7 + h_8 = 0. \quad (7.7)$$

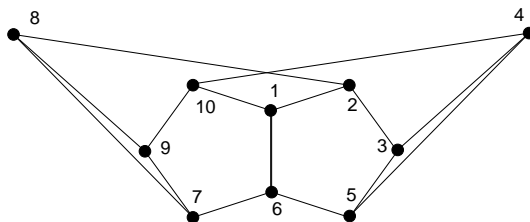
This shows that the embedding introduces a relation among the generators of the Cartan subalgebra. Constructing the quotient algebra

$$\mathfrak{g}_{(10_3, 10_3)_4} = \frac{(\mathbb{KM}(A(\mathfrak{g}_{(10_3, 10_3)_4})))'}{\mathfrak{r}}, \quad (7.8)$$

corresponds to eliminating one of the generators  $h_a$  through eq. (7.7). It is that quotient algebra that is dual to the geometric configuration  $(10_3, 10_3)_4$ . Note that the roots are not



**Figure 24:**  $(10_3, 10_3)_4$ : This configuration gives rise to a Dynkin diagram whose Cartan matrix has vanishing determinant, and hence contains a non-trivial center.



**Figure 25:** The Dynkin diagram of  $(10_3, 10_3)_4$  corresponds to a Cartan matrix with vanishing determinant and hence to an algebra that contains a non-trivial ideal.

linearly independent but obeys the same linear relation as the Cartan elements in eq. (7.7).

Note that that the Kac-Moody algebra associated with the degenerate Cartan matrix eq. (7.3) along the lines of [18] involves augmenting the Cartan matrix to get a “realization” and adding one more Cartan generator. The metric in the complete Cartan subalgebra of this Kac-Moody algebra has signature  $(-, -, +, +, \dots, +)$ .

### 7.3 Dynkin diagrams dual to configurations $(10_3, 10_3)$

We now give, in the form of a table, the list of all configurations  $(10_3, 10_3)$  and the corresponding Dynkin diagrams. These are all connected. Note that some of the configurations give rise to equivalent Dynkin diagrams. For instance, the configuration  $(10_3, 10_3)_2$  and the Desargues configuration  $(10_3, 10_3)_3$  (which is projectively self-dual) both lead to the Petersen Dynkin diagram. Thus, although we have ten configurations, we only find seven distinct rank 10 subalgebras of  $E_{10}$  : six Lorentzian subalgebras and one subalgebra with a Cartan matrix having zero determinant. The degenerate case discussed above appears for two of the configurations,  $(10_3, 10_3)_4$  and  $(10_3, 10_3)_7$ . All other cases give Cartan matrices with one negative and nine positive eigenvalues. Only the first configuration,  $(10_3, 10_3)_1$ ,



is non-planar, i.e. cannot be realized with straight lines in the plane [36]. This fact does not seem to manifest itself on the algebraic side.

Since some of the configurations give rise to equivalent Dynkin diagrams one might wonder if this means that two cosmological solutions may seem different from the supergravity point of view but are in fact equivalent in the coset construction. This is not true because even though the Dynkin diagrams are the same, the embedding in  $E_{10}$  is not. Hence, when constructing a coset Lagrangian based on the algebra associated to a given configuration, one must choose the generators according to the numbering of the lines in the configurations, and this uniquely determines the solution. This is also motivated by the use of the word “dual” for the correspondences we find.

## 8. Conclusions

In this paper, we have shed a new algebraic light on previous work on M-theory cosmology [1]. This has been done by associating to each geometric configuration  $(n_m, g_3)$  a regular, electric subalgebra of  $E_n$ , through the following rule: the line incidence diagram of the geometric configuration  $(n_m, g_3)$  is the Dynkin diagram of the corresponding regular subalgebra of  $E_n$ . We have also derived explicitly which subalgebras arise for all known geometric configurations with  $n \leq 10$ . In this context, a particularly intriguing case was the realization of the Petersen graph as the Dynkin diagram of a rank-10 Lorentzian subalgebra of  $E_{10}$ .

These somewhat unexpected mathematical results encompass other cosmological solutions besides those given in [1], since the algebras that we have exhibited underlie many interesting, time-dependent M-theory solutions. In particular, we found that  $\sigma$ -models for commuting  $A_1$ -subalgebras of  $E_{10}$  give rise to intersecting  $SM2$ -brane solutions [37]. This result is similar in spirit to that of [38, 39] where it was discovered, in the context of  $\mathfrak{g}^{+++}$ -algebras, that the intersection rules for  $Mq$ -branes are encoded in orthogonality conditions between the various roots of  $\mathfrak{g}^{+++}$ . These intersection rules apply also to spacelike branes [40] so they are of interest for some of the solutions discussed in this paper. For two  $Sq$ -branes,  $A$  and  $B$ , in  $M$ -theory the rules are [41, 42]

$$SMq_A \cap SMq_B = \frac{(q_A + 1)(q_B + 1)}{9} - 1. \tag{8.1}$$

So, for example, if we have two  $SM2$ -branes the result is

$$SM2 \cap SM2 = 0, \tag{8.2}$$

which means that they are allowed to intersect on a 0-brane. Note that since we are dealing with spacelike branes, a 0-brane is extended in one spatial direction so the two  $SM2$ -branes may therefore intersect in one spatial direction only. Hence, the intersection rules are fulfilled for the relevant configurations, namely  $(3_1, 1_3)$ ,  $(6_2, 4_3)$  and  $(7_3, 7_3)$ . So, in our treatment the orthogonality conditions of [38] are equivalent to only exciting commuting  $A_1$ -algebras<sup>8</sup>, regularly embedded in  $E_{10}$ . This implies that the intersection rules are

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<sup>8</sup>This was also pointed out in [33].

	Configuration	Dynkin diagram	Determinant of $A$
$(10_3, 10_3)_1$			$\det A(\mathfrak{g}_{(10_3, 10_3)_1}) = -121$
$(10_3, 10_3)_2$			$\det A(\mathfrak{g}_{(10_3, 10_3)_2}) = -256$
$(10_3, 10_3)_3$			$\det A(\mathfrak{g}_{(10_3, 10_3)_3}) = -256$
$(10_3, 10_3)_4$			$\det A(\mathfrak{g}_{(10_3, 10_3)_4}) = 0$
$(10_3, 10_3)_5$			$\det A(\mathfrak{g}_{(10_3, 10_3)_5}) = -16$

**Table 4:**  $n = 10$  configurations and their dual Lorentzian Kac-Moody algebras. Note that some of the configurations give rise to equivalent Dynkin diagrams. In this table and in the next one, we have drawn the Dynkin diagrams in a way that minimizes the crossing number (i.e., the unwanted crossings of edges at points that do not belong to the Dynkin diagram).

automatically fulfilled for configurations with no parallel lines.

Our paper can be developed in various directions. One can look for explicit new

	Configuration	Dynkin diagram	Determinant of $A$
$(10_3, 10_3)_6$			$\det A(\mathfrak{g}_{(10_3, 10_3)_6}) = -16$
$(10_3, 10_3)_7$			$\det A(\mathfrak{g}_{(10_3, 10_3)_7}) = 0$
$(10_3, 10_3)_8$			$\det A(\mathfrak{g}_{(10_3, 10_3)_8}) = -64$
$(10_3, 10_3)_9$			$\det A(\mathfrak{g}_{(10_3, 10_3)_9}) = -49$
$(10_3, 10_3)_{10}$			$\det A(\mathfrak{g}_{(10_3, 10_3)_{10}}) = -25$

**Table 5:**  $n = 10$  configurations and their dual Lorentzian Kac-Moody algebras. Note that some of the configurations give rise to equivalent Dynkin diagrams. Here, we have ceased to number the points of the geometrical configurations as this information is not needed in order to draw the Dynkin diagram.

solutions using the sigma-model insight, for which techniques have been developed. Systems with  $n \leq 8$  are in principle integrable so here we know that solutions can be found in

closed form. However, also for  $n = 9, 10$ , simplifications should arise since the algebras are affine or Lorentzian. In the Lorentzian case the algebras are not hyperbolic so the associated cosmological solutions must be non chaotic, and hence explicit solutions should exist. Work along these lines is in progress. One might also perhaps get new information on the meaning of the higher level fields and the dictionary between supergravity and the sigma-model in the context of these simpler algebras.

Another interesting possibility is to extend the approach taken in this paper and consider “magnetic algebras” (for which the simple roots are all magnetic) and their associated configurations. This corresponds to exciting a set of fields at level 2 in the  $E_{10}$ -decomposition. The simplest case would be to consider a configuration with one line through six points. A possible choice of generators is

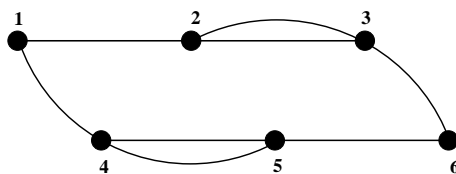
$$\begin{aligned}
 e &= E^{123456} & f &= F_{123456} \\
 h &= [E^{123456}, F_{123456}] = -\frac{1}{6} \sum_{a \neq 1, \dots, 6} K^a + \frac{1}{3}(K^1_1 + \dots + K^6_6).
 \end{aligned}
 \tag{8.3}$$

These generators constitute an  $A_1$ -subalgebra of  $E_{10}$  and the gravitational solution is precisely the  $SM5$ -brane solution of [37], i.e., in the billiard language, a bounce against a magnetic wall. In a sense this gives the simplest case of a duality between configurations with 3 points and 6 points. This can be seen as a manifestation of electric-magnetic duality from an algebraic point of view. An alternative approach could be to realize the magnetic algebras as configurations with four points on each line, corresponding to the spatial indices of the dual field strength, i.e. in the example above we would then associate a configuration to  $F_{789(10)}$  instead of  $A_{123456}$ .

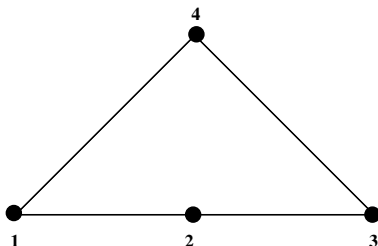
A natural line of development is to further consider configurations with  $n > 10$  since the association between geometric configurations and regular subalgebras holds for the whole  $E_n$  family. For instance, there exist 31 configurations of type  $(11_3, 11_3)$  [17] and these lead by our rules to rank-11 regular subalgebras of  $E_{11}$  that are either of Lorentzian type or, if their Cartan matrix is degenerate, of indefinite type with Cartan subalgebra embeddable in a space of Lorentzian signature. It would be of interest to investigate the solutions of eleven-dimensional supergravity to which these algebras give rise.

One might wonder if new features arise if we relax some of the rules defined in section 2.2. For example, the condition of  $m$  lines through each point was imposed mainly for aesthetical reasons since this gives interesting configurations. Relaxing it increases the number of different configurations for each  $n$ . These also lead to regular subalgebras of  $E_{10}$ . For instance, the set of 10 points with lines  $(123)$ ,  $(145)$ ,  $(167)$ ,  $(189)$  and  $(79(10))$  yields the algebra  $A_3 \oplus A_1 \oplus A_1$ , which is decomposable. However, these configurations with an unequal number of lines through each point seem to give rise to Dynkin diagrams with less structure.

We could also consider going beyond regular subalgebras of  $E_{10}$ . A way to achieve this is to relax Rule 3, i.e. that two points in the configuration determine at most one line. Let us check this for a simple example. For  $n = 6$  a possible configuration that violates Rule 3 is the set of six points and four lines discussed in [1] and shown in figure 26. In this case it is not interesting to define the generators as we have previously done since this



**Figure 26:** This set of six points, four lines containing three points each, with two lines through each point, is not a geometric configuration because it violates Rule 3: two points may determine more than one line.



**Figure 27:** The Dynkin diagram of  $A_3^+$ , associated with the above set of points and lines through the rules outlined in the text.

would give a non-sensical Cartan matrix. For example, defining  $e_1 = E^{123}$  and  $e_2 = E^{236}$  gives commutators of the form  $[h_1, e_2] = e_2$  and  $[h_2, e_1] = e_1$  which give rise to a Cartan matrix with positive entries. Instead, a reasonable choice of generators is

$$e_1 = E^{456} \quad e_2 = E^{123} \quad e_3 = F_{236} \quad e_4 = F_{145}. \tag{8.4}$$

These yield the Cartan matrix of the affine extension  $A_3^+$  of  $A_3$  (see figure 27). However, a new feature arises here because for example the commutator  $[e_1, e_4]$  does not give a level 2 generator but instead we find an off-diagonal level 0 generator

$$[e_1, e_4] = [E^{456}, F_{145}] = K^6_1. \tag{8.5}$$

In the gravitational solution this generator turns on an off-diagonal metric component and so it corresponds to going outside of the diagonal regime investigated in this paper (unless one imposes further conditions as in [1]). We further see that the embedding into  $E_{10}$  is not regular since positive root generators of the subalgebra are in fact negative root generators of  $E_{10}$ , and vice versa.

Finally, on the mathematical side, we have uncovered seven rank-10 Coxeter groups that are subgroups of the Weyl group of  $E_{10}$  and have the following properties:

- Their Coxeter graphs (which coincide with the Dynkin diagrams of the corresponding  $E_{10}$ -subalgebras in this “simply-laced” case) are connected.
- The only Coxeter exponents are 2 and 3 (i.e., the off-diagonal elements in the Cartan matrix are either 0 or -1).

- Each node is connected to exactly three other nodes.

Note that the determinants of their Cartan matrices are all minus squared integers<sup>9</sup>. It would be of interest to determine the automorphisms of the corresponding Lie algebras and investigate whether their embedding in  $E_{10}$  is maximal. Also, we have found two explicitly different embeddings for some of them and one might inquire whether they are equivalent. Investigations of these questions are currently in progress.

We also observe that the association of subalgebras of the relevant Kac-Moody algebras  $A_{D-3}^{++}$  to homogeneous cosmological models has been done in [43] in the context of pure gravity in spacetime dimensions  $D \leq 5$ .

## Acknowledgments

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<sup>9</sup>Let us mention that among the conceivable connected 10-points Coxeter graphs (with each node linked to three others), there are only four other cases with (non-positive) determinants: 0,-125,-165 and -192. These have only one negative eigenvalue. One might wonder if they define also Coxeter subgroups of the Weyl group of  $E_{10}$ .

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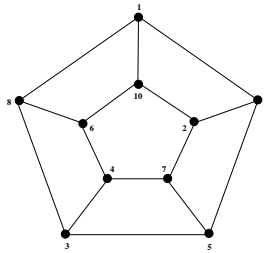


**Erratum**

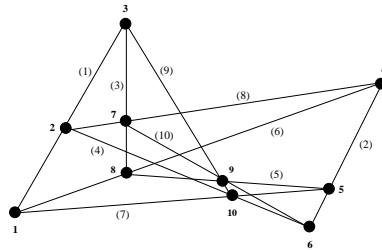
- In the third paragraph of section 5.3 the second sentence should be replaced by:

“Because the Killing form on the Cartan subalgebra of  $A_1$  is positive definite, one cannot find a solution of the Hamiltonian constraint if one turns on only  $A_1$ .”

- We further point out here that the Dynkin diagram corresponding to the configuration  $(10_3, 10_3)_1$  lacks a node in table 4 and should of course take the following form:



- In table 4, the geometric configuration  $(10_3, 10_3)_2$  was inadvertently incorrectly reproduced. The correct configuration is:



Note that our numbering of geometric configurations differ from that of reference [1] only in this case:  $(10_3, 10_3)_2$  here is  $(10_3, 10_3)_3$  of [1] whereas  $(10_3, 10_3)_3$  here is  $(10_3, 10_3)_2$  of [1].

- Finally, the configuration  $(10_3, 10_3)_3$  lacks numbering of the rightmost line and should be displayed as follows:

